Evaluating the performance of modified Anderson–Darling goodness-of-fit tests for normality

Damian Stoltmann,[a] Piotr Sulewski[b]

Abstract. The main aim of the article is to define and practically apply the modified Anderson–Darling (MAD) goodness-of-fit tests for normality. The modifications consist in varying the formula for calculating the empirical distribution function (EDF). Additional contributions of the paper include the expansion of the EDF family with four new proposals and the creation of a family of alternative distributions, consisting of both older and newer distributions that belong to all groups of skewness and kurtosis signs thanks to their flexibility. Critical values are obtained using 10^6 order statistics for sample sizes of $n = 10, 20$ and at a significance level of $\alpha = 0.05$. Finally, the article shows the calculation of the power of the analysed tests for alternative distributions based on 10^5 values of the test statistics. Their parameters have been selected to show several similarities to the normal distribution. The effectiveness of the tests is illustrated through the analysis of real datasets.

Keywords: empirical distribution function, goodness-of-fit test, Anderson–Darling test, test power

JEL: C14, C15

1. Introduction

A variety of goodness-of-fit tests (GoFTs) have been considered and applied in many fields of science. GoFTs for normality are very popular in economics and finance, where they are used to analyse market behaviour (the distribution of rates of return, trading volume or asset prices), assess market efficiency and identify deviations from ideal market conditions, analyse stochastic processes (asset prices or changes in commodity prices). Other examples of scientific areas where the application of GoFTs is common is demography, where the fertility curve is almost normally distributed, and econometrics, where normality tests are used to check whether regression errors are normally distributed. This is particularly important for the proper evaluation of regression models, as violating the assumption of normality can lead to erroneous statistical conclusions. In hydrology, specialists are interested in estimating flood magnitudes for high return periods. In these cases, however, the interest is focused on the upper (right) tail of the distribution. Moreover, one of the most important problems in hydrology is

[a] Pomeranian University in Słupsk, Institute of Exact and Technical Sciences, Department of Computer Science, ul. Arciszewskiego 22a, 76-200 Słupsk, Poland, e-mail: damian.stoltmann@upsl.edu.pl, ORCID: https://orcid.org/0000-0001-7053-2684.

[b] Pomeranian University in Słupsk, Institute of Exact and Technical Sciences, Department of Computer Science, ul. Arciszewskiego 22a, 76-200 Słupsk, Poland, e-mail: piotr.sulewski@apsl.edu.pl, ORCID: https://orcid.org/0000-0002-0788-6567.

the estimation of the quantile for a specific return period. Then the goodness of fit in the upper or lower (left) tail of the distribution is more important than the fit of the entire region (Ma et al., 2024).

One of the most common testing procedures available in statistical software is the Anderson-Darling (AD) test (Anderson & Darling, 1952), which belongs to a group of tests based on the empirical distribution function (EDF). Other popular EDF tests include the Kolmogorov-Smirnov (KS) test (Kolmogoroff, 1933; Smirnov, 1948), the Lilliefors (LF) test (Lilliefors, 1967), the Cramér-von Mises (CVM) test (Cramér, 1928; von Mises 1931), the Kuiper (K) test (Kuiper, 1960), and the Watson (W) test (Watson, 1962).

Recently, many articles have been devoted to GoFTs for normality. Table 1 shows the works created in the 21st century and their authors.

Table 1. Articles devoted to normal GoFTs written in the 21st century

Article	Sample sizes	Article	Sample sizes
Bonett and Seier (2002)	$\mathbf{10}, \mathbf{20}, \dots, \mathbf{50}, 100$	Afeez et al. (2018)	$\mathbf{10}, \mathbf{30}, \mathbf{50}, 100, \dots$
Aliaga et al. (2003)	-	Marange and Qin (2019)	$\mathbf{15}, \mathbf{30}, \mathbf{50}, 80, \dots$
Bontemps and Meddahi (2005)	$100,250,500, \dots$	Sulewski (2019)	$\mathbf{10}, \mathbf{12}, \dots, \mathbf{30}, \mathbf{40}, \mathbf{50}$
Luceño (2006)	100	Tavakoli et al. (2019)	$\mathbf{5}, \mathbf{6}, \dots, \mathbf{15}, \mathbf{20}, \mathbf{25}, \mathbf{30}, \mathbf{40}, \mathbf{50} \dots ,100$
Yazici and Yolacan (2007)	$\mathbf{20}, \mathbf{30}, \mathbf{40}, \mathbf{50}$	Mishra et al. (2019)	$\boldsymbol{n} < \mathbf{30}, \boldsymbol{n} > \mathbf{30}$
Gel at el. (2007)	$\mathbf{20}, \mathbf{50}, 100$	Kellner and Celisse (2019)	$\mathbf{50}, 75,100,200, \dots$
Coin (2008)	$\mathbf{20}, \mathbf{50}, 200$	Wijekularathna et al. (2020)	$\mathbf{5}, \mathbf{10}, \mathbf{20}, \mathbf{30}, \mathbf{50}, 75, \dots$
Brys et al. (2008)	$100,1000$	Sulewski (2022a)	$\mathbf{10}, \mathbf{14}, \mathbf{20}$
Gel and Gastwirth (2008)	$\mathbf{30}, \mathbf{50}, 100$	Hernandez (2021)	$\mathbf{5}, \mathbf{10}, \dots, \mathbf{30}$
Romão et al. (2010)	$\mathbf{25}, \mathbf{50}, 100$	Khatun (2021)	$\mathbf{10}, \mathbf{20}, \mathbf{25}, \mathbf{30}, \mathbf{40}, \mathbf{50}, 100,200,300$
Razali and Wah (2011)	$\mathbf{20}, \mathbf{30}, \mathbf{50}, 100, \dots$	Arnastauskaitė et al. (2021)	$\mathbf{2^5}, 2^6, \dots, 2^{10}$
Noughabi and Arghami (2011)	$\mathbf{10}, \mathbf{20}, \mathbf{30}, \mathbf{50}$	Bayoud (2021)	$\mathbf{10}, \mathbf{20}, \dots, \mathbf{50}, 60,80,100$
Yap and Sim (2011)	$\mathbf{10}, \mathbf{20}, \mathbf{30}, \mathbf{50}, 100, \dots$	Uhm and Yi (2023)	$\mathbf{10}, \mathbf{20}, \mathbf{30}, 100,200,300$
Chernobai et al. (2005)	-	Sulewski (2021)	$\mathbf{20}, \mathbf{50}, 100$
Ahmad and Khan (2015)	$\mathbf{10}, \mathbf{20}, \dots, \mathbf{50}, 100, \dots$	Desgagne et al. (2023)	$\mathbf{20}, \mathbf{50}, 100,200$
Mbah and Paothong (2015)	$\mathbf{10}, \mathbf{20}, \mathbf{30}, \mathbf{50}, 100, \dots$	Uyanto (2022)	$\mathbf{10}, \mathbf{30}, \mathbf{50}, 70,100$
Torabi et al. (2016)	$\mathbf{10}, \mathbf{20}, \mathbf{50}, 100, \dots$	Ma et al. (2024)	$\mathbf{10}, \mathbf{30}, \mathbf{50}$
Feuerverger (2016)	200	Giles (2024)	$\mathbf{10}, \mathbf{25}, \mathbf{50}, 100, 250 ,500,1000$
Nosakhare and Bright (2017)	$\mathbf{5}, \mathbf{10}, \dots, \mathbf{50}, 100$	Borrajo et al. (2024)	$\mathbf{50}, 100,200,500$
Desgagné and Lafaye de Micheaux (2018)	$\mathbf{10}, \mathbf{12}, \dots, \mathbf{20}, \mathbf{50}, 100, ..$	Terán-García and Pérez-Fernández (2024)	$\mathbf{25}, 900$

Note. Sample sizes of $n \leq 50$ are in bold.
Source: authors' work.

The small samples that dominate in Table 1 are common in experimental economics, where the research described in the published articles was based on samples of a dozen or so people in a group.

This is where strong tests can be particularly useful, as situations may happen that although the hypothesis is accepted in the original article, the hypothesis is rejected when a stronger test is applied.

From a methodological perspective, this paper is situated within the literature on nonparametric goodness-of-fit testing, and in particular within the class of EDF-based tests for normality. Unlike approaches based on parametric modeling, EDF-based tests such as the KS, CVM and AD tests rely solely on the comparison between the empirical and theoretical distribution functions and therefore remain fully nonparametric under the null hypothesis.

The proposed modified Anderson-Darling (MAD) tests belong to this class and contribute to the existing literature by systematically investigating alternative definitions of the EDF within the Anderson-Darling framework. Special emphasis is placed on small sample sizes which dominate many practical applications and a substantial part of the existing literature, as shown in Table 1. In particular, the analysis focuses on small sample sizes where classical asymptotic approximations may be unreliable and where the improvements in test power are especially relevant.

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be independent and identically distributed observations from unknown continuous cumulative distribution function (CDF) $F(x)$. We wish to know whether $F(x)$ coincides with a CDF of the normal distribution $\Phi(x)$. We are interested in testing hypothesis $H_0: F(x) = \Phi(x)$ against $H_1: F(x) \neq \Phi(x)$. The EDF is given by $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \theta(x - x_i)$, where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$.

The δ-corrected KS test (Harter et al., 1984), further investigated by Khamis (1990, 1992, 1993), redefines the value of the EDF at the data points and compares the redefined EDF to the CDF at the data points. Let the EDF at the i-th data point be given by

$$F_\delta(x) = \frac{i-\delta}{n-2\delta+1}, 0 \leq \delta \leq 1. \tag{1}$$

Harter et al. (1984) selected $\delta = 0,0.5,1$ for study.

Bloom (1958) proposed the α, β transformation:

$$F_{\alpha,\beta}\left(x_{(i)}\right) = \frac{i-\alpha}{n-\alpha-\beta+1}, \alpha, \beta \leq 1 \tag{2}$$

to decrease the MSE of certain statistics. Note that $F_{\delta,\delta}(x) = F_\delta(x)$. This transformation was used to create the GoFTs. Sulewski (2022a) used the Bloom formula to create the one-component Lilliefors GoFT with statistic

$$\overline{\overline{LF}} = \underbrace{max}_{i}\{|F_{\alpha,\beta}(x_{(i)}) - \Phi(x_{(i)})|\}. \quad (3)$$

It is commonly known that the greatest discrepancy between the theoretical and empirical distribution functions may occur at different positions in the series. The probability that this discrepancy appears at a given positional statistic r decreases as r becomes more extreme. The idea of a two-component test statistic is described in Sulewski (2021). The first component (denoted by $\overline{\overline{LF}}$) is, as in the original LF test, the absolute value of the greatest discrepancy between the sample and population distributions, while the second component (denoted by r) is the position in an ordered sample where this discrepancy is located. Both components, $\overline{\overline{LF}}$ and r, are random variables.

Simulation studies for the one- and two-component Lilliefors tests were carried out by Sulewski (2021, 2022a) for the following methods of calculating $F_{\alpha,\beta}(x_{(i)})$ $(\alpha,\beta \leq 1)$:

1. $F_{0,1}(x_{(i)}) = \frac{i}{n}$ – occurs in the KS statistic;
2. $F_{1,0}(x_{(i)}) = \frac{i-1}{n}$ – occurs in the KS statistic;
3. $F_{\frac{1}{2},\frac{1}{2}}(x_{(i)}) = \frac{i-0.5}{n}$ – occurs in the CM statistic;
4. $F_{0,0}(x_{(i)}) = \frac{i}{n+1}$ – the mean value of *i*-th order statistic of the beta distribution;
5. $F_{\frac{3}{10},\frac{3}{10}}(x_{(i)}) = \frac{i-0.3}{n+0.4}$ – the median of *i*-th order statistic of the beta distribution;
6. $F_{\frac{3}{8},\frac{3}{8}}(x_{(i)}) = \frac{i-0.375}{n+0.25}$ – the mean value of *i*-th order statistic of the normal distribution;
7. $F_{\frac{127}{400},\frac{127}{400}}(x_{(i)}) = \frac{i-0.3175}{n+0.365}$ – proposed by Filliben (1975);
8. $F_{1,1}(x_{(i)}) = \frac{i-1}{n-1}$ – proposed by Harter et al. (1984).

In six of the EDF definitions listed above (except $F_{0,1}$ and $F_{1,0}$) $\alpha = \beta$.

The main aim of the article is to define and practically apply modified Anderson–Darling (MAD) goodness-of-fit tests for normality. We propose an MAD goodness-of-fit test for normality in which the classical EDF is replaced by Bloom's (α,β) family of EDF estimators. In contrast to previous studies, Bloom's formula is applied using values of α and β that have not been previously investigated in the context of Anderson-Darling-type tests. Additional contributions of the study to the existing body of research involve the extension of the EDF family by four new EDF definitions, which are incorporated into the proposed modified Anderson-Darling framework. Moreover, rather than introducing new probability distributions, the paper constructs a structured family of alternative distributions by grouping the existing classical and modern distributions according to the signs of skewness and excess kurtosis, which allows a systematic and interpretable comparison of the test power. Finally, the power of the proposed tests is evaluated using Monte Carlo simulations based on $rep = 10^5$ replications, where each replication corresponds to a simulated value of the test statistic under a given alternative distribution. The critical values are obtained using simulated order statistics for sample sizes of $n = 10$ and $n = 20$ at the significance level of $\alpha = 0.05$.

2. Modified Anderson-Darling tests for normality

Let $z_{(i)} = \left(x_{(i)} - \bar{x}\right)/s$. Goodness-of-fit tests based on EDF form a broad and well-established class of procedures for assessing normality. Among them, the KS and CVM tests are the two fundamental approaches, differing primarily in the way the deviations between empirical distribution function F_n and the hypothesised distribution are aggregated.

The KS test belongs to the supremum class of EDF statistics and focuses on the maximum absolute deviation between F_n and the theoretical distribution. In contrast, the CVM test belongs to the quadratic EDF class and integrates squared deviations over the entire support of the distribution. The CVM statistic is given by

$$CVM = n\int_{-\infty}^{\infty}[F_n(z) - \Phi(z)]^2 d\Phi(z). \tag{4}$$

The computing formulas for the CVM statistic is given by

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n}\left[\Phi\left(z_{(i)}\right) - \frac{i-0.5}{n}\right]^2 = \frac{n}{3} + \frac{1}{n}\sum_{i=1}^{n}(1-2i)\Phi\left(z_{(i)}\right) + \sum_{i=1}^{n}\Phi\left(z_{(i)}\right)^2. \tag{5}$$

A general representation of the CVM family is given by

$$FAD = n\int_{-\infty}^{\infty}[F_n(z) - \Phi(z)]^2 \omega\left(\Phi(z)\right) d\Phi(z), \tag{6}$$

where $\omega\left(\Phi(z)\right): [0,1] \to R^+$ is a weight function.

The AD statistic corresponds to weight function

$$\omega\left(\Phi(z)\right) = \{\Phi(z)[1 - \Phi(z)]\}^{-1}, \tag{7}$$

placing greater emphasis on the tails of the distribution. The classical AD statistic is defined as

$$AD = n\int_{-\infty}^{\infty}\frac{[F_n(z) - \Phi(z)]^2}{\Phi(z)[1 - \Phi(z)]} d\Phi(z). \tag{8}$$

The lower and upper tail AD statistics marked AD^L and AD^U are defined as (Sinclair et al., 1990)

$$AD^L = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{\Phi(z)} d\Phi(z),\ AD^U = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{1-\Phi(z)} d\Phi(z). \quad (9)$$

Note that $AD = AD^L + AD^U$. The computing formulas for the AD, AD^L and AD^U statistics are as follows:

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)ln[\Phi(z_{(i)})] + (2n-2i+1)ln[1-\Phi(z_{(i)})]\}, \quad (10)$$

$$AD^L = -\frac{3n}{2} + 2\sum_{i=1}^{n}\Phi(z_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)ln[\Phi(z_{(i)})], \quad (11)$$

$$AD^U = \frac{n}{2} - 2\sum_{i=1}^{n}\Phi(z_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2n-2i+1)ln[1-\Phi(z_{(i)})]. \quad (12)$$

The lower-tail and upper-tail AD statistics measure the discrepancies between the empirical and theoretical distributions that are concentrated in the lower and upper tails, respectively, rather than over the entire distribution. Such tail-focused statistics are particularly useful in applications where departures from normality occur mainly in the extremes and where small sample sizes limit the power of global goodness-of-fit tests.

Stephens (1974, 1979) proposed modifications of the AD statistics denoted as AD_1, AD_2 and given by

$$\overline{AD} = AD\left(1 + \frac{4}{n} - \frac{25}{n^2}\right), \overline{\overline{AD}} = AD\left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right). \quad (13)$$

The AD test is the member of the CVM family for normality.

It is important to replace the weight function in order to raise the power of the test statistic. (7) denotes AD statistic (8) and $\omega(\Phi(z)) = 1$ corresponds to the CVM statistic (Cramér, 1928; von Mises, 1931).

Rodriguez and Viollaz (1995) considered $\omega(\Phi(z)) = [2\Phi(z) - \Phi(z)^2]^{-1}$ for the lower tail and $\omega(\Phi(z)) = [1 - \Phi(z)^2]^{-1}$ for the upper tail of the distribution. Thus, the next members of the CVM family are

$$ADR^L = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{2\Phi(z)-\Phi(z)^2} d\Phi(z), ADR^U = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{1-\Phi(z)^2} d\Phi(z), \text{ respectively.} \quad (14)$$

The computing formulas for the ADR_L and ADR_U statistics are as follows:

$$ADR^L = n[ln(4) - 1] + \frac{\sum_{i=1}^{n}(2i-4n-1)ln[2-\Phi(z_{(i)})] - \sum_{i=1}^{n}(2i-1)ln[\Phi(z_{(i)})]}{2n}, \quad (15)$$

$$ADR^U = n[ln(4) - 1] + \frac{\sum_{i=1}^{n}(2i-2n-1)ln[1-\Phi(z_{(i)})] - \sum_{i=1}^{n}(2i+2n-1)ln[1+\Phi(z_{(i)})]}{2n}. \quad (16)$$

Luceño (2006) proposed $\omega(\Phi(z)) = \Phi(z)^{-2}$ and $\omega(\Phi(z)) = [1-\Phi(z)]^{-2}$ for the lower and upper tails of the distribution, respectively. The modified AD statistics marked as ADL^L and ADL^U are as follows:

$$ADL^L = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{\Phi(z)^2} d\Phi(z), ADL^M = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{[1-\Phi(z)]^2} d\Phi(z). \quad (17)$$

The statistic for the entire distribution is given by Luceño (2006):

$$ADL = ADL_L + ADL_U. \quad (18)$$

The computing formulas for the ADL^L, ADL^U and ADL statistics are given by Chernobai et al. (2005 and Luceño (2006):

$$ADL^L = 2\sum_{i=1}^{n} ln[\Phi(z_{(i)})] + \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{\Phi(z_{(i)})}, \quad (19)$$

$$ADL^U = 2\sum_{i=1}^{n} ln[1-\Phi(z_{(i)})] + \frac{1}{n}\sum_{i=1}^{n}\frac{2n-2i+1}{1-\Phi(z_{(i)})}, \quad (20)$$

$$ADL = 2ln\{\prod_{i=1}^{n}\Phi(z_{(i)})[1-\Phi(z_{(i)})]\} + \frac{1}{n}\sum_{i=1}^{n}\left[\frac{2i-1}{\Phi(z_{(i)})} + \frac{2n-2i+1}{1-\Phi(z_{(i)})}\right]. \quad (21)$$

Ma et al. (2024) proposed $\omega(\Phi(z)) = \Phi(z)^{-1.5}$ and $\omega(\Phi(z)) = [1-\Phi(z)]^{-1.5}$ for the lower and upper tails of the distribution, respectively. The modified AD statistics marked as ADM^L and ADM^U are as follows:

$$ADM^L = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{\Phi(z)^{1.5}} d\Phi(z), ADM^U = n\int_{-\infty}^{\infty} \frac{[F_n(z)-\Phi(z)]^2}{[1-\Phi(z)]^{1.5}} d\Phi(z). \quad (22)$$

The computing formulas for the ADM^L and ADM^U statistics are defined as

$$ADM^L = 2\sum_{i=1}^{n}\left[2\sqrt{\Phi\left(z_{(i)}\right)} + \frac{2i-1}{n\sqrt{\Phi\left(z_{(i)}\right)}}\right] - \frac{16}{3}n, \tag{23}$$

$$ADM^U = 2\sum_{i=1}^{n}\left[2\sqrt{1-\Phi\left(z_{(i)}\right)} + \frac{2n-2i+1}{n\sqrt{1-\Phi\left(z_{(i)}\right)}}\right] - \frac{16}{3}n. \tag{24}$$

Feuerverger (2016) considered $\omega\left(\Phi(z)\right) = [1-\Phi(z)]^{-\vartheta}$ $(0 \le \vartheta < 2)$ for the upper tail of the distribution. Thus, the next member of the CVM family is as follows:

$$ADF^U(v) = n\int_{-\infty}^{\infty}\frac{[F_n(z)-\Phi(z)]^2}{[1-\Phi(z)]^v}d\Phi(z)\ (0 \le \vartheta \le 2). \tag{25}$$

Note that $ADF^U(1) = AD^U$, $ADF^U(2) = ADC^U$, $ADF^U(1.5) = ADM^U$, $ADF^U(0) = CVM$. The computing formula for the $ADF_U(v)$ statistic has the form of:

$$ADF^U(\vartheta) = C(n,v) + \frac{2}{2-\vartheta}\sum_{i=1}^{n}\left[1-\Phi\left(z_{(i)}\right)\right]^{2-v} + \frac{\sum_{i=1}^{n}(2n-2i+1)\left[1-\Phi\left(z_{(i)}\right)\right]^{1-\vartheta}}{n(v-1)}, \tag{26}$$

where $C(n,v) = \frac{n}{3-\vartheta} - \frac{2n}{2-\vartheta} + \frac{n}{1-\vartheta}$.

There is a second version of the AD test for normality – the variance-weighted KS test marked as $|AD|$, which belongs to the supremum class (Chernobai et al., 2005). The $|AD|$ statistic is given by

$$|AD| = \sqrt{n}max\left\{\underbrace{sup}_{i}\left[\frac{\frac{i}{n}-\Phi\left(z_{(i)}\right)}{\sqrt{\Phi\left(z_{(i)}\right)\left[1-\Phi\left(z_{(i)}\right)\right]}}\right], \underbrace{sup}_{i}\left[\frac{\Phi\left(z_{(i)}\right)-\frac{i-1}{n}}{\sqrt{\Phi\left(z_{(i)}\right)\left[1-\Phi\left(z_{(i)}\right)\right]}}\right]\right\}. \tag{27}$$

The lower and upper tails $|AD|$ statistics marked as $|AD_L|$ and $|AD_U|$ are defined as

$$|AD^L| = \sqrt{n}max\left\{\underbrace{sup}_{i}\left[\frac{\frac{i}{n}-\Phi\left(z_{(i)}\right)}{\Phi\left(z_{(i)}\right)}\right], \underbrace{sup}_{i}\left[\frac{\Phi\left(z_{(i)}\right)-\frac{i-1}{n}}{\Phi\left(z_{(i)}\right)}\right]\right\}, \tag{28}$$

$$|AD^U| = \sqrt{n} max \left\{ \underbrace{sup}_{i} \left[\frac{\frac{i}{n} - \Phi(z_{(i)})}{1 - \Phi(z_{(i)})} \right], \underbrace{sup}_{i} \left[\frac{\Phi(z_{(i)}) - \frac{i-1}{n}}{1 - \Phi(z_{(i)})} \right] \right\}. \quad (29)$$

All computing formulas were verified with appropriate integral formulas in Mathcad by comparing the closed-form results with a numerical integration for a range of sample sizes and order statistics.

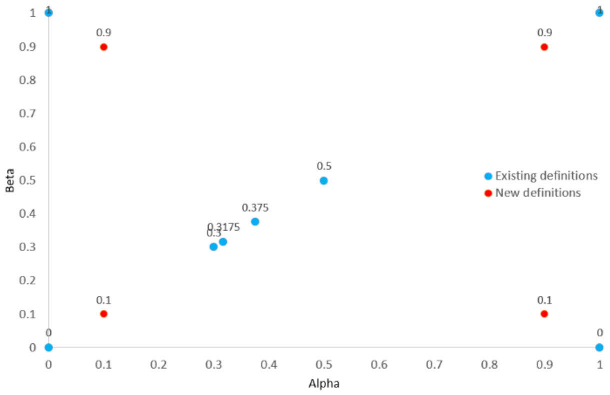
3. New modified Anderson-Darling test for normality

Before we present the MAD test, we expand the EDF family with four new proposals: $F_{0.1,0.1}, F_{0.9,0.1}, F_{0.9,0.9}$ and $F_{0.1,0.9}$, given by

$$F_{\frac{1}{10},\frac{1}{10}}(z_{(i)}) = \frac{i-0.1}{n+0.8}, F_{\frac{9}{10},\frac{1}{10}}(z_{(i)}) = \frac{i-0.9}{n}, F_{\frac{9}{10},\frac{9}{10}}(z_{(i)}) = \frac{i-0.9}{n-0.8}, F_{\frac{1}{10},\frac{9}{10}}(z_{(i)}) = \frac{i-0.1}{n}. \quad (30)$$

Thus, eight of the EDF definitions listed earlier are on the $\beta = \alpha$ line and five of them are on the $\beta = -\alpha + 1$ line (see Figure 1). The previously analysed values unevenly fill the $\beta = \alpha$ line over interval $[0, 1]$. Four values belong to the $[0.3, 0.5]$ interval. Value 0.1 represents interval $(0, 0.3)$ and the values of 0.9 represent interval $(0.5, 1)$. The new representatives of $\beta = -\alpha + 1$ line, also located at the corners of the square, are EDFs with $\alpha = 0.9, \beta = 0.1$ and $\alpha = 0.1, \beta = 0.9$.

Figure 1. Graphical representation of EDF definitions



Source: authors' work.

The computing formula for the AD statistic (10) can be written as

$$AD = -n - 2 \sum_{i=1}^{n} \left\{ \frac{i-0.5}{n} ln[\Phi(z_{(i)})] + \frac{n-i+0.5}{n} ln[1 - \Phi(z_{(i)})] \right\} \quad (31)$$

or

$$AD = -n - 2\sum_{i=1}^{n}\left\{\frac{i-0.5}{n} ln\left[\Phi\left(z_{(i)}\right)\right] + \left(1 - \frac{i-0.5}{n}\right) ln\left[1 - \Phi\left(z_{(i)}\right)\right]\right\}. \quad (32)$$

The AD statistic can be expressed as a functional of the EDF. The commonly used computing formulas (31)-(32) correspond to the classical choice of the EDF given by

$$\hat{F}\left(x_{(i)}\right) = \frac{i-0.5}{n}.$$

Bloom (1958) proposed a two-parameter family of EDF estimators of the form of (2) which includes the classical EDF as a special case for $\alpha = \beta = 0.5$. Replacing $\hat{F}$ in the AD functional with $F_{\alpha,\beta}$ yields the following modified Anderson–Darling statistic:

$$AD_{\alpha,\beta} = -n - 2\sum_{i=1}^{n}\left\{F_{\alpha,\beta}\left(x_{(i)}\right) ln\left[\Phi\left(z_{(i)}\right)\right] + \left[1 - F_{\alpha,\beta}\left(x_{(i)}\right)\right] ln\left[1 - \Phi\left(z_{(i)}\right)\right]\right\}. \quad (33)$$

Note that $AD_{0.5,0.5} = AD$.

4. Similarity measure

Let's assume that

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \,, s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \,, z_{(i)} = \frac{\left(x_{(i)} - \bar{x}\right)}{s}, \quad (34)$$

$$m_k = \frac{1}{n}\sum_{i=1}^{n}\left(x_{(i)} - \bar{x}\right)^k, \gamma_1 = \frac{m_3}{s^3}, \bar{\gamma}_2 = \frac{m_4}{s^4} - 3. \quad (35)$$

Note that the Malachov inequality is defined as $\bar{\gamma}_2 \geq {\gamma_1}^2 - 2$ (Malachov, 1978).

A review of the recent statistical literature shows that the values of small skewness γ_1 and excess kurtosis $\bar{\gamma}_2$ do not dominate in testing for normality. It is very interesting to see how the GoFTs will respond to samples coming from alternatives close to normal distribution.

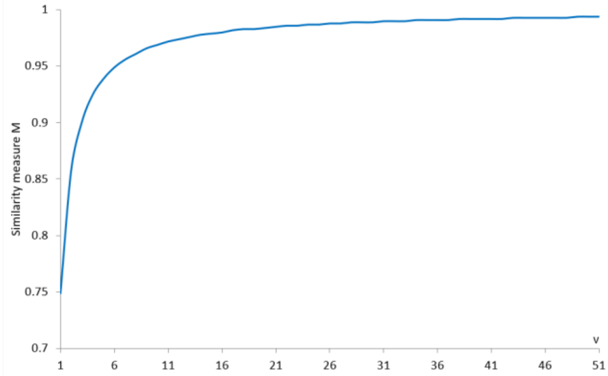
Let $f(x; \boldsymbol{\theta})$ be a PDF of the alternative with vector of parameters $\boldsymbol{\theta}$**.** Similarity measure M of the alternative (Alt) to the normal distribution is defined as (Sulewski, 2022a)

$$M(\boldsymbol{\theta}; \mu, \sigma) = \int_{-\infty}^{\infty} min[f(x; \boldsymbol{\theta}), \phi(x; \mu, \sigma)]dx, \quad (36)$$

where $\phi(x;\mu,\sigma)$ is the PDF of the normal distribution. $M(\boldsymbol{\theta};\mu,\sigma)$ takes on the values of $[0,1]$. $M(\boldsymbol{\theta};\mu,\sigma)=1$ when the PDFs are identical.

Figure 2 shows the values of similarity measure (36), when an alternative is the Student's *t* distribution with v degrees of freedom. Note that if $v\to+\infty$, then obviously $M_t(v;0,1)\to 1$.

Figure 2. Similarity measure $M_t(v;0,1)$ for Student's *t* distribution



Source: authors' work.

5. Alternative distributions

As mentioned before, there are numerous articles devoted to testing for normality. In these articles, many alternative distributions (alternatives) are used, including asymmetric and symmetric ones. Note that symmetric distributions with undefined γ_1 and $\bar{\gamma}_2$ are Cauchy and slash distributions (S), which are defined as

$$S=\frac{N(0,1)}{U},$$

where $N(0,1)$ denotes standard normal distribution and $U\sim Uniform(0,1)$.

According to the statistical literature, alternatives can be divided into four groups, depending on the support and shape of their densities (see e.g. Esteban et al., 2001; Torabi et al., 2016). These groups include symmetric alternatives with support $(-\infty,\infty)$, asymmetric alternatives with support $(-\infty,\infty)$, alternatives with support $(0,\infty)$ and alternatives with support $(0,1)$. Gan and Koehler (1990), Krauczi (2009) and Torabi et al. (2016) divided alternatives into five groups, namely: asymmetric short-tailed, asymmetric long-tailed, symmetric short-tailed, symmetric close to normal and symmetric long-tailed alternatives. Sulewski (2021, 2022a) divided alternatives into 12 groups (A1-F2) by their γ_1 and $\bar{\gamma}_2$ signs and bimodality. The author showed that bimodality of the alternative does not affect the choice of the EDF definition.

As stated before, one of the aims of our article is to provide a new division of alternatives. Based on the proposal presented in Sulewski and Stoltmann (2026), our idea is to divide the alternatives into nine groups according to their γ_1 and $\bar{\gamma}_2$ signs. Groups O-H are defined in Table 2.

Table 2. Groups of alternatives with signs of γ_1 and $\bar{\gamma}_2$

Group	γ_1	$\bar{\gamma}_2$
O	zero	zero
A	positive	positive
B	negative	positive
C	zero	positive
D	zero	negative
E	positive	negative
F	negative	negative
G	positive	zero
H	negative	zero

Source: authors' work.

The main criterion for selecting an alternative for the Monte Carlo simulation is that γ_1 and $\bar{\gamma}_2$ calculated for the alternative parameters belong to the O, A–H groups. This criterion is fulfilled by distributions defined in an infinite domain such as:

- the Edgeworth series (ES) with parameters γ_1 and $\bar{\gamma}_2$ as a monolithic distribution (Kendall & Stuart, 1968);
- the Pearson (P) distribution, with parameters γ_1 and $\bar{\gamma}_2$ as a monolithic distribution (Pearson, 1895);
- the normal mixture (NM) distribution with 5 parameters as a mixture of two normal distributions;
- the normal distribution with plasticising component (NDPC) (Sulewski, 2022b) with 6 parameters as a mixture of normal and non-normal distributions;
- the plasticising component mixture (PCM) with seven parameters (Sulewski, 2022a) as a mixture of two identical non-normal distributions that characterises multimodality.

In each group of alternatives, we consider four families of distributions and four values of similarity measure M (36), namely $M = 0.50, 0.75, 0.90$ and 0.95. This results in a total of 16 alternative distributions per group. An exception concerns the P distribution in group C, for which the similarity measure could not be defined for all parameter values.

The Appendix presents Tables 1A–5A with vectors of alternative parameter $\boldsymbol{\theta}$, mean μ_a, standard deviation σ_a, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M for the analysed alternatives. The skewness and excess kurtosis tend to zero, while the similarity measure tends to unity. Often, the mean tends to zero, and the standard deviation tends to unity, while the similarity measure tends to unity. PDF formulas and PDF curves (see Figures 1A–5A) for the alternative $\boldsymbol{\theta}$ values are also provided in the Appendix. As shown in Figure 1A, the ES distribution is not suitable for simulation studies due to the negative PDF values for groups D-H even though the normalisation condition is met. Figure 4A shows interesting bimodal shapes. Figures 2A and 3A show both unimodal and bimodal shapes. Figure 1A shows unimodal shapes, and Figure 5A presents very interesting multimodal shapes.

6. Power study

The use of lower and upper tail versions of the AD-type statistics is motivated by the fact that many practically relevant departures from normality are asymmetric and manifest primarily in one tail of the distribution. In such cases, global goodness-of-fit statistics may lose power, especially for small sample sizes, whereas tail-focused tests can detect localised deviations in the lower or upper tail more effectively. This approach is consistent with the earlier findings reported in Chernobai et al. (2005) and is particularly relevant for the grouped alternatives considered in this study.

The use of Bloom's formula (2) in the one-component (Sulewski, 2022a) and two-component (Sulewski, 2021) LF statistic significantly influenced the power of tests (PoTs). Can we expect a similar situation to occur for the new modified AD test?

In Hernandez (2021), a sample of the most recent comparisons (since 1990) has been used to rank 55 different normality tests. The overall winner of this analysis is the regression-based Shapiro-Wilk (SW) test of normality.

29 GoFTs were selected for the simulation studies (see Table 3). New proposals numbered 19-29 were compared with LF, CVM, SW tests (Shapiro & Wilk, 1965) numbered 1–3 and AD modifications numbered 4–18 listed in Section 1. To study the power of each of the discussed tests, critical values $cv_{0.05}$ (the type I error $\alpha = 0.05$) were calculated using $m = 10^6$ simulated order statistics. The PoTs were calculated based on $rep = 10^5$ Monte Carlo replications for each alternative distribution and each value of the similarity measure. Table 3 shows the critical values (CVs) used in the simulation study and the test sizes (TSs) for the analysed tests. The names of the new modifications are in bold. In total, the simulation design combines 29 tests, 4 families of alternative distributions, 4 values of the similarity measure and two sample sizes.

Table 3. Critical values (CV) and test sizes (TS) of the analysed GoFTs for normality

No.	GoFT	CV		TS		No.	GoFT	CV		TS	
		n=10	n=20	n=10	n=20			n=10	n=20	n=10	n=20
1	LF	0.262	0.192	0.050	0.050	16	ADL	6.931	12.685	0.050	0.051
2	CVM	0.120	0.123	0.051	0.051	17	ADM^L	0.891	1.075	0.052	0.051
3	SW	0.844	0.904	0.050	0.050	18	ADM^U	0.889	1.078	0.050	0.051
4	$\lvert AD^L \rvert$	18.200	37.830	0.051	0.050	19	$\boldsymbol{AD_{0,1}}$	0.693	0.726	0.051	0.049
5	$\lvert AD^U \rvert$	18.235	38.061	0.050	0.050	20	$\boldsymbol{AD_{1,0}}$	0.693	0.726	0.050	0.050
6	$\lvert AD \rvert$	2.697	3.632	0.051	0.051	21	$\boldsymbol{AD_{0,0}}$	1.479	1.610	0.052	0.051
7	AD^L	6.931	12.685	0.050	0.051	22	$\boldsymbol{AD_{1,1}}$	-0.281	-0.261	0.052	0.051
8	AD^U	0.354	0.374	0.051	0.051	23	$\boldsymbol{AD_{0.3,0.3}}$	1.022	1.087	0.052	0.051
9	AD	0.687	0.721	0.052	0.051	24	$\boldsymbol{AD_{3/8,3/8}}$	0.900	0.952	0.052	0.051

10	$\overline{AD}$	0.790	0.821	0.052	0.051	25	$\boldsymbol{AD_{\frac{127}{400},\frac{127}{400}}}$	0.994	1.056	0.052	0.051
11	$\overline{\overline{AD}}$	0.754	0.753	0.052	0.051	26	$\boldsymbol{AD_{0.1,0.1}}$	1.333	1.439	0.052	0.051
12	ADR^L	0.216	0.226	0.052	0.051	27	$\boldsymbol{AD_{0.9,0.9}}$	-0.070	-0.056	0.052	0.051
13	ADR^U	0.216	0.227	0.052	0.051	28	$\boldsymbol{AD_{0.9,0.1}}$	0.691	0.724	0.051	0.050
14	ADL^L	3.510	6.256	0.051	0.051	29	$\boldsymbol{AD_{0.1,0.9}}$	0.691	0.725	0.051	0.049
15	ADL^U	3.515	6.289	0.051	0.051						

Source: authors' work.

The complete simulation results with power values fill a table with 29 columns (29 tests) and 32 rows (4 alternatives, 2 sample sizes, 4 values of a single similarity measure). Presenting such large tables is difficult due to the size of the article. Therefore, the conclusions apply to the full results, and only the most interesting results are shown in Tables 6–13. Alternatives are indexed, i.e. the larger the index, the more the distribution resembles a normal distribution (e.g. index 4 denotes the similarity measure of 0.95). The highest values in the row are in bold.

Of course, it is expected that the power of the GoFTs will increase as the sample size increases, and decrease as the value of similarity measure (36) increases. The following analysed tests do not meet these basic assumptions: $|AD^L|$ (groups A, D, E, F, G), $|AD^U|$ (B, D, E, F, H), $|AD|$ (D, E, F, G, H), AD^L (D, E, F, H), ADL^L (A, D, E, F, G), ADL^U (B, D, E, F, H), ADM^L for (A, D, E, G), ADM^U (D F, H) and ADM (D).

Powers of the $\overline{AD}$ and $\overline{\overline{AD}}$ tests are the same for all the analysed cases.

The average PoT is the highest for group B of the alternatives, followed by group A. This means that these GoFTs best detect samples from asymmetric distributions with positive excess kurtosis. The most problematic case, as might be expected, is detecting samples from symmetric distributions. The GoFTs best detect samples from the P distribution and the worst from the NM distribution.

The power of the modified AD tests for normality with $\alpha = \beta$ is very similar for all groups of alternatives. Tables 4 and 5 show exemplary results for groups A and H at significance level $\alpha = 0.05$. It is noteworthy that one test dominates for all the analysed alternatives and similarity measures. The $AD_{1,0}$ test is recommended for groups A, E and G. The $AD_{0,1}$ test is recommended for groups B, F and H. The $AD_{1,1}$ and $AD_{0,0}$ tests are recommended for groups C and D, respectively.

Table 4. The power of MAD tests for group A of alternatives (Alts)

Alt	n	MAD test										
		19	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**
P_1	10	0.764	**0.844**	0.817	0.814	0.816	0.816	0.816	0.817	0.814	0.839	0.775
P_2		0.231	**0.340**	0.291	0.291	0.291	0.291	0.291	0.291	0.291	0.331	0.243
P_3		0.082	**0.131**	0.106	0.107	0.106	0.106	0.106	0.106	0.107	0.126	0.087
P_4		0.064	**0.085**	0.073	0.074	0.073	0.073	0.073	0.073	0.074	0.082	0.066
P_1	20	0.990	**0.995**	0.993	0.993	0.993	0.993	0.993	0.993	0.993	0.995	0.990
P_2		0.535	**0.650**	0.599	0.599	0.599	0.599	0.599	0.599	0.599	0.642	0.549
P_3		0.146	**0.213**	0.179	0.183	0.180	0.181	0.180	0.180	0.183	0.208	0.153

P_4		0.080	**0.112**	0.097	0.099	0.097	0.098	0.097	0.097	0.099	0.109	0.084
NM_1	10	0.078	**0.126**	0.102	0.104	0.103	0.103	0.103	0.102	0.103	0.122	0.084
NM_2		0.074	**0.123**	0.097	0.098	0.098	0.098	0.098	0.097	0.098	0.117	0.078
NM_3		0.053	**0.074**	0.062	0.062	0.062	0.062	0.062	0.062	0.062	0.070	0.055
NM_4		0.049	**0.054**	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.054	0.050
NM_1	20	0.138	**0.207**	0.169	0.172	0.170	0.170	0.170	0.169	0.172	0.198	0.142
NM_2		0.133	**0.199**	0.167	0.169	0.168	0.168	0.168	0.167	0.169	0.195	0.141
NM_3		0.065	**0.092**	0.079	0.080	0.079	0.079	0.079	0.079	0.080	0.091	0.068
NM_4		0.051	0.057	0.055	0.056	0.055	0.055	0.055	0.055	0.056	**0.059**	0.052
$NDPC_1$	10	0.286	**0.381**	0.335	0.338	0.336	0.336	0.336	0.335	0.338	0.372	0.295
$NDPC_2$		0.103	**0.152**	0.130	0.129	0.130	0.130	0.130	0.130	0.129	0.148	0.108
$NDPC_3$		0.053	**0.059**	0.056	0.057	0.056	0.056	0.056	0.056	0.057	0.059	0.054
$NDPC_4$		0.052	**0.052**	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.052
$NDPC_1$	20	0.615	**0.697**	0.661	0.666	0.662	0.663	0.662	0.662	0.665	0.694	0.626
$NDPC_2$		0.198	**0.255**	0.230	0.229	0.230	0.229	0.229	0.230	0.229	0.254	0.206
$NDPC_3$		0.055	**0.063**	0.059	0.060	0.059	0.059	0.059	0.059	0.060	0.063	0.057
$NDPC_4$		0.052	**0.052**	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.051
PCM_1	10	0.559	**0.607**	0.594	0.588	0.592	0.592	0.592	0.594	0.588	0.605	0.567
PCM_2		0.162	**0.261**	0.217	0.218	0.217	0.217	0.217	0.217	0.218	0.252	0.173
PCM_3		0.054	**0.076**	0.065	0.066	0.065	0.065	0.065	0.065	0.066	0.074	0.056
PCM_4		0.051	**0.056**	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.055	0.051
PCM_1	20	0.879	**0.909**	0.896	0.894	0.895	0.895	0.895	0.896	0.894	0.908	0.883
PCM_2		0.412	**0.531**	0.481	0.481	0.481	0.481	0.481	0.481	0.482	0.522	0.426
PCM_3		0.065	**0.096**	0.082	0.083	0.082	0.082	0.082	0.082	0.083	0.094	0.068
PCM_4		0.052	**0.059**	0.055	0.056	0.056	0.056	0.056	0.056	0.056	0.058	0.052

Note. The highest test power is in bold.
Source: authors' work.

Table 5. The power of MAD tests for group H of alternatives (Alts)

Alt	n		MAD test									
		19	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**
P_1	10	**0.797**	0.724	0.769	0.762	0.767	0.767	0.767	0.768	0.763	0.735	0.792
P_2		**0.270**	0.175	0.228	0.225	0.227	0.227	0.227	0.228	0.225	0.186	0.262
P_3		**0.096**	0.056	0.078	0.077	0.078	0.078	0.078	0.078	0.078	0.060	0.091
P_4		**0.068**	0.047	0.057	0.057	0.057	0.057	0.057	0.057	0.057	0.049	0.066
P_1	20	**0.987**	0.980	0.986	0.985	0.985	0.985	0.985	0.985	0.985	0.981	0.987
P_2		**0.557**	0.446	0.507	0.501	0.505	0.505	0.505	0.506	0.501	0.459	0.547
P_3		**0.154**	0.099	0.126	0.124	0.126	0.125	0.125	0.126	0.125	0.105	0.149
P_4		**0.083**	0.054	0.070	0.070	0.070	0.070	0.070	0.070	0.070	0.057	0.080
NM_1	10	**0.099**	0.060	0.081	0.081	0.081	0.081	0.081	0.081	0.081	0.065	0.097
NM_2		**0.079**	0.051	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.054	0.077
NM_3		**0.054**	0.048	0.051	0.051	0.051	0.051	0.051	0.051	0.051	0.048	0.054
NM_4		**0.052**	0.049	0.050	0.051	0.050	0.051	0.050	0.050	0.051	0.050	0.052
NM_1	20	**0.156**	0.102	0.128	0.127	0.128	0.128	0.128	0.128	0.127	0.106	0.148
NM_2		**0.110**	0.071	0.091	0.091	0.092	0.091	0.092	0.092	0.091	0.075	0.106
NM_3		**0.056**	0.049	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.049	0.056
NM_4		**0.053**	0.050	0.051	0.050	0.051	0.051	0.051	0.051	0.050	0.049	0.052
$NDPC_1$	10	**0.142**	0.087	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.093	0.137

$NDPC_2$		**0.084**	0.053	0.067	0.066	0.066	0.066	0.066	0.066	0.066	0.056	0.081
$NDPC_3$		**0.053**	0.048	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.048	0.052
$NDPC_4$		**0.052**	0.051	0.051	0.051	0.051	0.051	0.051	0.051	0.051	0.051	0.051
$NDPC_1$	20	**0.253**	0.179	0.218	0.218	0.218	0.218	0.218	0.218	0.218	0.187	0.246
$NDPC_2$		**0.120**	0.075	0.098	0.097	0.098	0.097	0.098	0.098	0.098	0.080	0.115
$NDPC_3$		**0.055**	0.049	0.053	0.052	0.052	0.052	0.052	0.052	0.052	0.050	0.054
$NDPC_4$		**0.052**	0.051	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.051	0.052
PCM_1	10	**0.068**	0.048	0.057	0.057	0.057	0.057	0.057	0.057	0.057	0.050	0.066
PCM_2		**0.057**	0.047	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.048	0.056
PCM_3		**0.063**	0.050	0.057	0.057	0.057	0.057	0.057	0.057	0.057	0.051	0.062
PCM_4		**0.061**	0.050	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.051	0.060
PCM_1	20	**0.083**	0.057	0.069	0.068	0.068	0.068	0.068	0.069	0.068	0.059	0.080
PCM_2		**0.060**	0.047	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.049	0.059
PCM_3		**0.074**	0.058	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.060	0.073
PCM_4		**0.066**	0.052	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.054	0.064

Note. The highest test power is in bold.
Source: authors' work.

Table 6. The power of GoFTs for group A of alternatives (Alts)

Alt	n	GoFT									
		2	3	5	8	12	15	18	**20**	**28**	**27**
P_1	10	0.787	0.847	0.513	0.681	**0.856**	0.602	0.651	0.844	0.839	0.814
P_2		0.271	0.316	0.302	0.250	0.300	**0.322**	0.301	0.340	0.331	0.291
P_3		0.102	0.111	0.165	0.108	0.100	**0.167**	0.145	0.131	0.126	0.107
P_4		0.071	0.076	0.108	0.076	0.069	**0.108**	0.095	0.085	0.082	0.074
P_1	20	0.987	**0.997**	0.689	0.960	0.997	0.824	0.927	0.995	0.995	0.993
P_2		0.545	**0.671**	0.450	0.502	0.620	0.512	0.549	0.650	0.642	0.599
P_3		0.164	0.209	0.252	0.184	0.162	**0.269**	0.256	0.213	0.208	0.183
P_4		0.091	0.108	0.153	0.103	0.086	**0.159**	0.147	0.112	0.109	0.099
NM_1	10	0.100	0.106	0.160	0.106	0.095	**0.164**	0.142	0.126	0.122	0.103
NM_2		0.095	0.100	0.146	0.098	0.093	**0.150**	0.130	0.123	0.117	0.098
NM_3		0.061	0.062	0.090	0.062	0.061	**0.090**	0.078	0.074	0.070	0.062
NM_4		0.052	0.052	**0.060**	0.053	0.051	0.060	0.058	0.054	0.054	0.052
NM_1	20	0.158	0.185	0.224	0.180	0.147	**0.249**	0.246	0.207	0.198	0.172
NM_2		0.160	0.178	0.189	0.169	0.152	0.214	**0.218**	0.199	0.195	0.169
NM_3		0.077	0.083	0.108	0.082	0.074	**0.113**	0.108	0.092	0.091	0.080
NM_4		0.055	0.057	0.071	0.057	0.053	**0.072**	0.067	0.057	0.059	0.056
$NDPC_1$	10	0.338	0.318	0.292	0.349	0.302	0.344	0.368	**0.380**	0.372	0.338
$NDPC_2$		**0.135**	0.121	0.119	0.120	0.130	0.130	0.128	**0.152**	0.148	0.129
$NDPC_3$		0.057	0.054	0.062	0.056	0.057	**0.063**	0.061	0.060	0.059	0.057
$NDPC_4$		0.052	0.052	0.054	0.052	0.052	**0.054**	0.053	0.053	0.053	0.053
$NDPC_1$	20	0.661	0.608	0.349	**0.700**	0.584	0.504	0.674	0.700	0.694	0.665
$NDPC_2$		**0.245**	0.186	0.116	0.207	0.227	0.138	0.180	0.258	0.254	0.229
$NDPC_3$		0.062	0.054	0.059	0.060	0.059	0.062	**0.062**	0.064	0.063	0.060
$NDPC_4$		0.051	0.053	0.054	0.051	0.051	**0.056**	0.054	0.052	0.052	0.052
PCM_1	10	0.580	0.594	0.424	0.572	0.581	0.492	0.551	0.607	0.605	0.588
PCM_2		0.202	0.227	0.232	0.214	0.200	**0.258**	0.257	0.261	0.252	0.218
PCM_3		0.064	0.067	**0.099**	0.067	0.062	0.099	0.085	0.076	0.074	0.066

PCM_4		0.054	0.054	0.063	0.055	0.053	**0.063**	0.060	0.056	0.055	0.054
PCM_1	20	0.871	**0.896**	0.711	0.866	0.883	0.772	0.838	0.909	0.908	0.894
PCM_2		0.436	0.495	0.258	0.483	0.428	0.366	**0.496**	0.531	0.522	0.482
PCM_3		0.077	0.093	0.134	0.085	0.076	**0.139**	0.126	0.096	0.094	0.083
PCM_4		0.055	0.058	0.074	0.057	0.054	**0.075**	0.069	0.059	0.058	0.056

Source: authors' work.

Table 7. The power of GoFTs for group B of alternatives (Alts)

Alt	n	GoFT									
		2	3	4	8	13	14	17	19	**29**	**22**
P_1	10	0.785	0.846	0.510	**0.869**	0.855	0.600	0.645	0.846	0.841	0.811
P_2		0.270	0.315	0.302	0.306	0.300	**0.321**	0.297	0.340	0.331	0.290
P_3		0.101	0.112	0.166	0.098	0.100	**0.168**	0.144	0.130	0.125	0.107
P_4		0.071	0.074	**0.107**	0.069	0.070	**0.107**	0.094	0.083	0.081	0.074
P_1	20	0.986	0.997	0.691	**0.997**	0.996	0.826	0.927	0.995	0.995	0.993
P_2		0.547	**0.674**	0.454	0.635	0.621	0.516	0.553	0.650	0.640	0.602
P_3		0.166	0.207	0.251	0.159	0.162	**0.266**	0.254	0.215	0.209	0.183
P_4		0.091	0.108	0.154	0.083	0.085	**0.160**	0.147	0.110	0.107	0.099
NM_1	10	0.130	0.138	0.191	0.122	0.125	**0.197**	0.176	0.161	0.156	0.137
NM_2		0.102	0.105	0.135	0.102	0.103	**0.140**	0.124	0.125	0.123	0.105
NM_3		0.078	0.078	0.091	0.081	0.081	**0.094**	0.085	0.093	0.091	0.079
NM_4		0.055	0.056	0.077	0.054	0.055	**0.077**	0.068	0.063	0.062	0.056
NM_1	20	0.228	0.258	0.284	0.204	0.212	0.318	**0.323**	0.279	0.271	0.247
NM_2		0.175	0.174	0.159	0.168	0.170	0.180	**0.193**	0.206	0.200	0.178
NM_3		0.113	0.109	0.097	**0.117**	0.117	0.106	0.110	0.132	0.127	0.113
NM_4		0.063	0.065	0.092	0.059	0.059	**0.096**	0.090	0.073	0.072	0.064
$NDPC_1$	10	0.652	0.660	0.451	0.660	**0.664**	0.536	0.582	0.701	0.694	0.660
$NDPC_2$		0.353	0.366	0.372	0.328	0.337	**0.414**	0.408	0.408	0.401	0.366
$NDPC_3$		0.129	0.145	0.206	0.121	0.124	**0.211**	0.187	0.164	0.159	0.138
$NDPC_4$		0.051	0.050	0.062	0.051	0.051	**0.062**	0.057	0.054	0.054	0.051
$NDPC_1$	20	0.954	0.950	0.626	0.955	0.958	0.778	0.900	0.968	0.967	0.961
$NDPC_2$		0.660	0.692	0.561	0.613	0.631	0.672	**0.736**	0.715	0.709	0.687
$NDPC_3$		0.219	0.283	0.331	0.200	0.208	**0.365**	0.357	0.286	0.279	0.250
$NDPC_4$		0.052	0.056	0.069	0.053	0.053	**0.070**	0.066	0.059	0.058	0.054
PCM_1	10	0.214	0.224	0.263	0.197	0.203	**0.283**	0.269	0.260	0.253	0.224
PCM_2		**0.189**	0.177	0.165	**0.189**	**0.190**	0.181	0.177	0.217	0.212	0.186
PCM_3		0.057	0.059	**0.069**	0.058	0.058	**0.069**	0.065	0.061	0.061	0.058
PCM_4		0.052	0.054	**0.062**	0.053	0.053	**0.061**	0.059	0.056	0.056	0.054
PCM_1	20	0.422	0.442	0.367	0.377	0.393	0.447	**0.503**	0.486	0.478	0.447
PCM_2		**0.373**	0.307	0.186	0.357	0.364	0.225	0.283	0.388	0.382	0.359
PCM_3		0.064	0.078	**0.096**	0.064	0.064	**0.096**	0.090	0.073	0.072	0.069
PCM_4		0.056	0.063	**0.079**	0.055	0.055	0.077	0.074	0.062	0.061	0.058

Source: authors' work.

Table 8. The power of GoFTs for group C of alternatives (Alts)

Alt	n	GoFT									
		2	6	7	15	16	**22**	**27**	9	10	11
P_1	10	0.115	0.147	**0.148**	0.115	**0.148**	0.122	0.121	0.120	0.120	0.120
P_2		0.096	**0.122**	0.122	0.100	**0.122**	0.101	0.101	0.100	0.100	0.100

P_3		0.077	**0.097**	0.096	0.084	0.096	0.080	0.080	0.079	0.079	0.079
P_4		0.063	**0.072**	0.072	0.066	**0.072**	0.064	0.064	0.063	0.063	0.063
P_1	20	0.176	0.248	**0.253**	0.175	**0.253**	0.194	0.193	0.191	0.191	0.191
P_2		0.134	0.199	**0.203**	0.147	**0.203**	0.149	0.149	0.146	0.146	0.146
P_3		0.096	0.146	**0.148**	0.115	**0.148**	0.106	0.105	0.104	0.104	0.104
P_4		0.067	0.095	**0.095**	0.083	**0.095**	0.072	0.072	0.071	0.071	0.071
NM_1	10	**0.214**	0.165	0.174	0.140	0.174	0.202	0.202	0.200	0.200	0.200
NM_2		0.130	0.129	0.131	0.110	0.131	0.129	0.128	0.127	0.127	0.127
NM_3		0.064	**0.076**	0.076	0.071	**0.076**	0.065	0.065	0.065	0.065	0.065
NM_4		0.058	**0.064**	0.063	0.060	**0.063**	0.059	0.059	0.059	0.059	0.059
NM_1	20	**0.391**	0.204	0.225	0.178	0.225	0.361	0.361	0.357	0.357	0.357
NM_2		**0.214**	0.160	0.170	0.141	0.170	0.207	0.206	0.204	0.204	0.204
NM_3		0.072	0.101	**0.102**	0.087	**0.102**	0.077	0.076	0.076	0.076	0.076
NM_4		0.059	0.075	**0.076**	0.071	**0.076**	0.061	0.061	0.061	0.061	0.061
$NDPC_1$	10	**0.179**	0.125	0.133	0.115	0.133	0.169	0.169	0.168	0.168	0.168
$NDPC_2$		0.060	0.067	**0.067**	0.063	**0.067**	0.062	0.062	0.061	0.061	0.061
$NDPC_3$		0.052	**0.054**	0.054	0.052	**0.054**	0.052	0.052	0.052	0.052	0.052
$NDPC_4$		0.051	0.053	**0.053**	0.051	**0.053**	0.051	0.051	0.051	0.051	0.051
$NDPC_1$	20	**0.323**	0.098	0.116	0.108	0.116	0.297	0.297	0.294	0.294	0.294
$NDPC_2$		0.063	0.079	**0.080**	0.074	**0.080**	0.065	0.065	0.065	0.065	0.065
$NDPC_3$		0.054	0.059	**0.059**	0.057	**0.059**	0.054	0.054	0.054	0.054	0.054
$NDPC_4$		0.050	**0.051**	**0.051**	**0.052**	**0.051**	0.050	0.050	0.050	0.050	0.050
PCM_1	10	0.082	0.095	**0.095**	0.084	**0.095**	0.085	0.084	0.083	0.083	0.083
PCM_2		**0.107**	0.095	0.097	0.087	0.097	0.102	0.102	0.101	0.101	0.101
PCM_3		0.077	**0.078**	**0.078**	0.073	**0.078**	0.075	0.075	0.074	0.074	0.074
PCM_4		0.059	**0.067**	0.067	0.064	**0.067**	0.060	0.060	0.060	0.060	0.060
PCM_1	20	0.107	0.108	0.112	0.102	**0.112**	0.111	0.111	0.110	0.110	0.110
PCM_2		**0.160**	0.104	0.109	0.099	0.109	0.146	0.145	0.144	0.144	0.144
PCM_3		**0.100**	0.084	0.086	0.081	0.086	0.095	0.095	0.094	0.094	0.094
PCM_4		0.061	0.086	**0.087**	0.077	**0.087**	0.065	0.065	0.065	0.065	0.065

Source: authors' work.

Table 9. The power of GoFTs for group D of alternatives (Alts)

Alt	n	GoFT									
		1	2	3	12	13	18	**21**	8	**26**	**23**
P_1	10	0.365	0.524	**0.667**	0.562	0.564	0.461	0.608	0.559	0.606	0.602
P_2		0.081	0.104	**0.127**	0.112	0.113	0.074	0.121	0.113	0.120	0.118
P_3		0.045	0.045	0.043	0.045	0.046	0.033	0.046	0.046	0.046	0.045
P_4		0.042	0.041	0.039	0.040	0.041	0.035	0.041	0.041	0.041	0.040
P_1	20	0.716	0.890	**0.980**	0.925	0.925	0.843	0.953	0.921	0.952	0.951
P_2		0.148	0.229	**0.352**	0.266	0.263	0.147	0.295	0.264	0.294	0.290
P_3		0.053	0.059	0.060	0.062	0.062	0.027	**0.065**	0.062	0.065	0.064
P_4		**0.045**	0.044	0.038	0.043	0.044	0.024	0.045	0.044	0.044	0.044
NM_1	10	0.163	0.194	0.179	0.173	**0.201**	0.123	0.199	0.198	0.197	0.194
NM_2		0.141	0.164	0.212	0.113	**0.266**	**0.304**	0.201	0.298	0.200	0.198
NM_3		**0.048**	0.048	0.045	0.047	**0.048**	0.039	0.048	0.048	0.048	0.047
NM_4		**0.048**	0.048	0.047	0.048	**0.048**	0.047	0.048	0.048	0.048	0.048
NM_1	20	0.364	0.481	0.418	0.418	**0.493**	0.256	0.486	0.485	0.484	0.479

NM_2		0.323	0.308	0.478	0.205	**0.556**	**0.598**	0.440	0.597	0.438	0.435
NM_3		0.059	0.062	0.055	0.060	0.061	0.032	0.063	0.061	0.062	0.062
NM_4		0.049	0.049	0.046	0.048	0.048	0.045	0.048	0.048	0.048	0.048
$NDPC_1$	10	0.099	0.110	0.101	0.106	0.108	0.064	0.112	0.105	0.111	0.110
$NDPC_2$		**0.053**	**0.053**	0.049	**0.053**	**0.053**	0.040	**0.053**	0.052	0.053	0.052
$NDPC_3$		**0.047**	0.046	0.045	0.046	**0.047**	0.045	0.046	0.047	0.046	0.046
$NDPC_4$		0.049	0.050	0.049	0.049	**0.050**	0.049	0.049	0.050	0.049	0.049
$NDPC_1$	20	0.188	0.242	0.203	0.228	0.227	0.092	0.243	0.217	0.241	0.238
$NDPC_2$		0.071	**0.071**	0.058	0.067	0.067	0.033	0.069	0.065	0.069	0.068
$NDPC_3$		**0.047**	0.046	0.040	0.045	0.045	0.038	0.044	0.045	0.044	0.044
$NDPC_4$		**0.049**	0.049	0.047	0.048	0.047	0.048	0.048	0.048	0.048	0.048
PCM_1	10	**0.046**	0.046	0.043	0.045	0.045	0.043	0.045	0.046	0.045	0.045
PCM_2		0.056	0.056	0.054	0.055	0.055	0.046	0.056	0.055	0.056	0.055
PCM_3		**0.048**	0.047	0.046	0.047	0.047	0.044	0.048	0.048	0.047	0.047
PCM_4		0.046	0.046	0.046	0.046	0.046	0.045	0.046	0.046	0.046	0.046
PCM_1	20	**0.047**	0.044	0.039	0.044	0.044	0.035	0.043	0.043	0.043	0.043
PCM_2		**0.068**	0.068	0.063	0.066	0.066	0.046	0.067	0.065	0.067	0.066
PCM_3		**0.050**	0.048	0.045	0.048	0.047	0.040	0.048	0.047	0.048	0.048
PCM_4		**0.049**	0.048	0.045	0.047	0.047	0.041	0.047	0.047	0.047	0.047

Source: authors' work.

Table 10. The power of GoFTs for group E of alternatives (Alts)

Alt	n	GoFT									
		1	2	3	5	8	12	13	15	**23**	18
P_1	10	0.577	0.698	0.775	0.240	0.566	**0.791**	0.616	0.345	0.737	0.471
P_2		0.149	0.188	0.224	0.132	0.144	**0.236**	0.159	0.147	0.205	0.148
P_3		0.065	0.069	0.074	0.096	0.067	**0.072**	0.068	**0.098**	0.071	0.085
P_4		0.052	0.053	0.054	0.074	0.054	0.052	0.054	**0.075**	0.053	0.065
P_1	20	0.902	0.961	**0.990**	0.166	0.900	0.988	0.930	0.369	0.980	0.760
P_2		0.297	0.402	**0.531**	0.106	0.293	0.529	0.330	0.148	0.461	0.231
P_3		0.096	0.106	**0.128**	0.109	0.098	0.121	0.101	0.119	0.115	0.120
P_4		0.060	0.062	0.066	0.083	0.063	0.062	**0.064**	**0.087**	**0.064**	0.082
NM_1	10	0.086	0.093	**0.094**	0.107	0.088	0.092	0.091	**0.113**	**0.094**	0.105
NM_2		0.063	0.064	**0.065**	0.089	0.065	0.064	**0.065**	**0.091**	**0.065**	0.081
NM_3		0.052	0.052	0.051	**0.066**	0.053	0.051	0.053	0.066	0.052	**0.060**
NM_4		0.051	0.051	0.050	0.053	0.051	0.050	0.051	**0.054**	0.051	0.053
NM_1	20	0.141	0.162	0.158	0.097	0.152	0.161	0.157	0.117	**0.165**	0.148
NM_2		0.080	0.086	0.084	0.083	0.086	0.082	0.087	0.093	0.087	**0.099**
NM_3		0.054	0.056	0.053	0.054	0.058	0.054	0.058	0.060	0.056	**0.065**
NM_4		0.051	0.050	0.051	0.056	0.050	0.051	0.050	**0.056**	0.050	0.054
$NDPC_1$	10	0.594	**0.659**	0.604	0.157	0.567	0.653	0.602	0.273	0.648	0.430
$NDPC_2$		0.072	**0.075**	0.073	0.063	0.070	**0.076**	0.073	0.068	**0.075**	0.071
$NDPC_3$		0.058	0.058	0.055	0.050	0.056	**0.058**	0.057	0.053	0.057	0.055
$NDPC_4$		**0.048**	0.047	0.045	0.047	0.046	0.047	0.046	0.047	0.046	0.047
$NDPC_1$	20	0.940	**0.974**	0.940	0.080	0.940	0.969	0.956	0.225	0.969	0.750
$NDPC_2$		0.113	0.129	0.114	0.031	0.117	0.128	0.122	0.041	0.129	0.078
$NDPC_3$		0.073	**0.078**	0.065	0.029	0.073	0.074	0.075	0.033	0.076	0.052
$NDPC_4$		**0.051**	0.051	0.046	0.038	0.048	0.051	0.049	0.039	0.050	0.043

PCM_1	10	0.104	0.112	0.117	0.058	0.088	**0.131**	0.095	0.064	0.116	0.074
PCM_2		0.066	0.069	0.065	0.042	0.062	**0.070**	0.064	0.045	0.068	0.054
PCM_3		0.051	0.052	0.050	0.049	0.053	0.052	0.053	0.051	0.052	**0.054**
PCM_4		0.049	0.048	0.046	0.040	0.047	0.049	0.047	0.041	0.048	0.044
PCM_1	20	0.208	0.226	0.256	0.040	0.154	**0.294**	0.174	0.046	0.245	0.085
PCM_2		0.102	0.114	0.097	0.026	0.094	**0.116**	0.100	0.028	0.112	0.051
PCM_3		0.062	0.068	0.064	0.022	**0.075**	0.064	0.074	0.027	0.070	0.052
PCM_4		**0.053**	0.051	0.045	0.027	0.048	0.051	0.049	0.028	0.050	0.035

Source: authors' work.

Table 11. The power of GoFTs for group F of alternatives (Alts)

Alt	n	GoFT									
		1	2	3	4	8	12	14	19	**29**	13
P_1	10	0.580	0.699	0.777	0.238	**0.813**	0.615	0.344	0.762	0.758	0.794
P_2		0.151	0.191	0.227	0.134	**0.252**	0.160	0.150	0.243	0.237	0.239
P_3		0.066	0.069	0.073	0.097	0.074	0.067	**0.099**	0.089	0.086	0.073
P_4		0.052	0.053	0.053	0.076	0.053	0.053	**0.076**	0.062	0.061	0.053
P_1	20	0.901	0.961	0.990	0.166	**0.991**	0.929	0.372	0.982	0.982	0.988
P_2		0.299	0.403	0.533	0.108	**0.557**	0.334	0.150	0.501	0.493	0.530
P_3		0.095	0.105	**0.127**	0.109	0.123	0.101	0.119	0.137	0.132	0.119
P_4		0.060	0.061	0.064	0.083	0.060	0.063	**0.086**	0.073	0.070	0.061
NM_1	10	0.219	0.256	0.241	0.106	0.250	0.240	0.141	0.291	0.283	0.254
NM_2		0.064	0.067	0.070	0.057	**0.080**	0.061	0.058	0.079	0.077	0.077
NM_3		0.050	0.050	0.049	0.055	0.049	0.049	**0.055**	0.052	0.052	0.049
NM_4		0.049	0.049	0.049	**0.051**	0.049	0.048	0.051	0.050	0.049	0.050
NM_1	20	0.477	0.592	0.533	0.049	0.557	0.560	0.097	0.616	0.609	0.572
NM_2		0.091	0.102	0.111	0.053	**0.139**	0.082	0.054	0.120	0.120	0.131
NM_3		0.051	0.050	0.049	0.055	0.048	0.050	**0.056**	0.055	0.051	0.048
NM_4		0.050	0.049	0.050	0.051	0.049	0.050	**0.051**	0.050	0.049	0.049
$NDPC_1$	10	0.199	**0.217**	0.192	0.098	0.207	0.197	0.125	0.240	0.234	0.211
$NDPC_2$		0.064	0.064	0.061	0.057	**0.065**	0.061	0.061	0.075	0.073	0.065
$NDPC_3$		0.049	0.048	0.046	0.051	0.046	0.048	**0.052**	0.050	0.049	0.047
$NDPC_4$		0.048	0.047	0.046	0.047	**0.048**	0.048	0.048	0.049	0.049	0.047
$NDPC_1$	20	0.437	**0.499**	0.394	0.051	0.449	0.453	0.088	0.501	0.495	0.464
$NDPC_2$		0.093	**0.100**	0.082	0.036	0.093	0.092	0.042	0.109	0.106	0.095
$NDPC_3$		0.048	0.045	0.044	**0.050**	0.045	0.046	0.050	0.048	0.047	0.044
$NDPC_4$		**0.050**	0.048	0.046	0.044	0.048	0.047	0.044	0.050	0.049	0.048
PCM_1	10	0.100	0.118	0.130	0.057	0.110	**0.133**	0.068	0.144	0.141	0.113
PCM_2		0.055	0.055	0.052	0.050	**0.059**	0.052	0.050	0.057	0.056	0.058
PCM_3		**0.047**	0.045	0.043	0.038	0.044	0.044	0.038	0.045	0.045	0.044
PCM_4		0.047	0.047	0.046	0.038	0.046	0.047	0.038	0.047	0.047	0.046
PCM_1	20	0.196	0.271	0.341	0.013	0.244	**0.343**	0.030	0.340	0.336	0.255
PCM_2		0.066	0.069	0.066	0.054	**0.087**	0.058	0.055	0.070	0.069	0.083
PCM_3		**0.046**	0.044	0.040	0.029	0.043	0.044	0.029	0.043	0.043	0.043
PCM_4		**0.052**	0.051	0.046	0.031	0.049	0.050	0.031	0.049	0.049	0.049

Source: authors' work.

Table 12. The power of GoFTs for group G of alternatives (Alts)

Alt	n	GoFT

		3	5	12	15	20	**28**	**21**	**26**	**23**	**25**
P_1	10	0.799	0.302	**0.818**	0.410	0.795	0.790	0.768	0.767	0.766	0.766
P_2		0.249	0.177	**0.256**	0.195	0.269	0.261	0.228	0.228	0.227	0.227
P_3		0.080	0.110	0.077	**0.112**	0.097	0.093	0.077	0.077	0.077	0.077
P_4		0.057	0.082	0.055	**0.083**	0.069	0.067	0.057	0.057	0.057	0.057
P_1	20	**0.992**	0.268	0.992	0.485	0.987	0.987	0.985	0.984	0.984	0.984
P_2		**0.585**	0.191	0.568	0.245	0.555	0.547	0.507	0.507	0.505	0.505
P_3		0.144	0.136	0.129	**0.150**	0.153	0.148	0.127	0.127	0.127	0.127
P_4		0.074	0.102	0.067	**0.106**	0.084	0.082	0.070	0.070	0.070	0.070
NM_1	10	0.065	0.093	0.063	**0.095**	0.081	0.076	0.064	0.064	0.064	0.064
NM_2		0.051	**0.066**	0.051	**0.065**	0.056	0.056	0.052	0.052	0.051	0.051
NM_3		0.051	0.061	0.050	**0.061**	0.054	0.054	0.051	0.051	0.051	0.051
NM_4		0.050	**0.056**	0.049	**0.055**	0.052	0.052	0.050	0.050	0.050	0.050
NM_1	20	0.094	0.104	0.088	**0.112**	0.112	0.107	0.092	0.092	0.092	0.092
NM_2		0.056	0.070	0.055	**0.072**	0.061	0.062	0.056	0.056	0.056	0.056
NM_3		0.053	0.064	0.052	**0.065**	0.057	0.056	0.052	0.052	0.052	0.052
NM_4		0.051	0.057	0.051	**0.057**	0.052	0.052	0.050	0.050	0.050	0.050
$NDPC_1$	10	0.125	0.111	**0.134**	0.117	0.148	0.144	0.124	0.124	0.124	0.124
$NDPC_2$		0.092	0.068	0.093	0.069	0.101	0.100	0.096	0.096	0.094	0.094
$NDPC_3$		0.067	0.091	0.067	**0.094**	0.083	0.080	0.069	0.069	0.069	0.069
$NDPC_4$		0.051	0.049	0.051	0.049	0.052	0.052	0.051	0.051	0.051	0.051
$NDPC_1$	20	0.247	0.122	**0.267**	0.138	0.269	0.263	0.240	0.240	0.239	0.239
$NDPC_2$		0.166	0.113	**0.175**	0.112	0.179	0.178	0.173	0.172	0.171	0.171
$NDPC_3$		0.088	0.088	0.089	0.099	0.112	0.109	0.094	0.094	0.094	0.094
$NDPC_4$		**0.057**	0.056	0.054	0.055	0.054	0.055	0.055	0.055	0.055	0.055
PCM_1	10	0.194	0.150	0.204	0.166	0.233	0.227	0.205	0.205	0.205	0.205
PCM_2		0.127	0.096	0.132	0.098	0.142	0.140	0.133	0.132	0.130	0.130
PCM_3		0.061	0.091	0.059	**0.092**	0.075	0.072	0.060	0.060	0.060	0.060
PCM_4		0.061	0.087	0.058	**0.089**	0.071	0.069	0.061	0.061	0.061	0.061
PCM_1	20	0.340	0.135	0.393	0.189	0.447	0.441	0.410	0.409	0.409	0.409
PCM_2		0.261	0.162	**0.273**	0.171	0.281	0.277	0.266	0.265	0.263	0.263
PCM_3		0.086	0.110	0.077	**0.118**	0.099	0.096	0.078	0.079	0.078	0.078
PCM_4		0.077	0.096	0.071	**0.105**	0.090	0.087	0.075	0.075	0.075	0.075

Source: authors' work.

Table 13. The power of GoFTs for group H of alternatives (Alts)

Alt	n	GoFT									
		4	8	14	17	19	2	**29**	3	13	21
P_1	10	0.303	**0.838**	0.412	0.513	0.797	0.732	0.792	0.801	0.819	0.769
P_2		0.178	**0.267**	0.194	0.185	0.270	0.209	0.262	0.250	0.255	0.228
P_3		0.111	0.079	**0.112**	0.096	0.096	0.075	0.091	0.081	0.079	0.078
P_4		0.082	0.055	**0.083**	0.072	0.068	0.056	0.066	0.058	0.055	0.057
P_1	20	0.267	**0.994**	0.486	0.801	0.987	0.972	0.987	0.993	0.992	0.986
P_2		0.192	**0.589**	0.246	0.316	0.557	0.445	0.547	0.584	0.565	0.507
P_3		0.137	0.130	**0.148**	0.144	0.154	0.116	0.149	0.143	0.127	0.126
P_4		0.102	0.066	**0.106**	0.098	0.083	0.068	0.080	0.075	0.066	0.070
NM_1	10	0.107	0.081	**0.111**	0.097	0.099	0.080	0.097	0.081	0.081	0.081
NM_2		0.096	0.065	**0.098**	0.085	0.079	0.064	0.077	0.066	0.064	0.066

NM_3		0.059	0.050	**0.059**	0.056	0.054	0.051	0.054	0.050	0.050	0.051
NM_4		0.055	0.051	**0.056**	0.053	0.052	0.051	0.052	0.050	0.050	0.050
NM_1	20	0.119	0.125	0.132	**0.139**	0.156	0.124	0.148	0.131	0.125	0.128
NM_2		0.102	0.084	0.112	**0.113**	0.110	0.089	0.106	0.092	0.085	0.091
NM_3		0.063	0.051	**0.065**	0.062	0.056	0.052	0.056	0.053	0.051	0.052
NM_4		0.057	0.049	**0.057**	0.056	0.053	0.050	0.052	0.051	0.049	0.051
$NDPC_1$	10	0.139	0.111	**0.148**	0.136	0.142	0.117	0.137	0.114	0.113	0.117
$NDPC_2$		0.102	0.067	**0.102**	0.087	0.084	0.064	0.081	0.068	0.066	0.067
$NDPC_3$		**0.057**	0.050	0.057	0.053	0.053	0.049	0.052	0.050	0.050	0.050
$NDPC_4$		0.051	0.051	0.051	**0.052**	0.052	0.051	0.051	0.050	0.051	0.051
$NDPC_1$	20	0.137	0.193	0.170	0.214	**0.253**	0.216	0.246	0.207	0.200	0.218
$NDPC_2$		0.122	0.098	**0.133**	0.129	0.120	0.088	0.115	0.111	0.096	0.098
$NDPC_3$		**0.067**	0.051	0.067	0.062	0.055	0.052	0.054	0.054	0.051	0.053
$NDPC_4$		**0.051**	0.050	0.051	0.051	0.052	0.050	0.052	0.050	0.050	0.050
PCM_1	10	0.082	0.056	**0.082**	0.071	0.068	0.056	0.066	0.057	0.056	0.057
PCM_2		**0.065**	0.052	0.065	0.060	0.057	0.052	0.056	0.053	0.052	0.053
PCM_3		0.065	0.059	**0.066**	0.061	0.063	0.058	0.062	0.056	0.058	0.057
PCM_4		0.067	0.054	**0.068**	0.062	0.061	0.054	0.060	0.054	0.054	0.054
PCM_1	20	0.092	0.065	**0.097**	0.091	0.083	0.067	0.080	0.070	0.066	0.069
PCM_2		0.071	0.053	**0.072**	0.068	0.060	0.054	0.059	0.056	0.053	0.055
PCM_3		0.064	0.066	0.067	0.067	**0.074**	**0.068**	0.073	0.063	0.067	0.066
PCM_4		0.062	0.057	0.066	**0.066**	0.066	0.060	0.064	0.055	0.058	0.059

Source: authors' work.

Tables 6–13 present the power of the top ten tests for groups A-H of the alternatives. The highest values in each row are in bold. The GoFTs that stand out for the groups of alternatives based on the sum of powers are as follows: $AD_{1,0}, AD_{0.9,0.1}, ADM^U$ (group A), $AD_{0,1}, AD_{0.1,0.9}, ADM^L$ (group B), $CVM, AD_{1,1}, AD_{0.9,0.9}$ (group C), $AD^U, ADR^U, AD_{0,0}$ (group D), $AD_{1,0}, AD_{0.9,0.1}, ADR^L$ (group E), $AD_{0,1}, AD_{0.1,0.9}, AD^U$ (group F), $AD_{1,0}, AD_{0.9,0.1}, ADR^L$ (group G) and $AD_{0,1}, AD_{0.1,0.9}$, SW (group H).

The sum of the powers for the groups of alternatives marked with indices 1 and 2 is the highest for $AD_{1,0}, AD_{0.9,0.1}, SW$ (group A), $AD_{0,1}, AD_{0.1,0.9}, SW$ (group B), $CVM, AD_{1,1}, AD_{0.9,0.9}$ (group C), AD^U, ADR^U, SW (group D), $AD_{1,0}, AD_{0.9,0.1}, ADR^L$ (groups E, G), $AD_{0,1}, AD_{0.1,0.9}, AD^U$ (groups F, H) tests.

The sum of the powers for the groups of alternatives marked with indices 3 and 4 is the highest for $ADL^U, |AD^U|, ADM^U$ (groups A, G), $ADL^L, |AD^L|, ADM^L$ (groups B, H), $AD^L, ADL, |AD|$ (group C), $AD_{0,0}, LF, AD_{0.1,0.1}$ (group D), $AD_{1,0}, AD_{0.9,0.1}, AD_{0,0}$ (group E), $AD_{0,1}, AD_{0.1,0.9}, ADL^L$ (group F) tests.

The modified AD tests with test statistics (33), for the given alternative and similarity measure, achieves the highest empirical power in 28%, 22%, 0%, 34%, 53%, 31%, 41%, 19% of the considered cases related to groups A–H, respectively.

The observed dominance of particular modified AD tests within specific skewness-kurtosis regimes can be explained by the manner in which Bloom's EDF parameters (α, β) control the relative weighting of discrepancies in the lower and upper tails of the distribution. Tests with $\alpha > \beta$ place greater emphasis on the upper tail, which increases their sensitivity to right-skewed alternatives or distributions with heavy upper tails. This explains the strong performance of tests such as $AD_{1,0}$ and $AD_{0.9,0.1}$ for groups characterised by positive skewness (groups A and G).

Conversely, tests where $\beta > \alpha$ emphasise deviations in the lower tail and therefore exhibit a higher power for left-skewed alternatives, as observed for $AD_{0,1}$ and $AD_{0.1,0.9}$ in groups B, F and H. For symmetric alternatives, the dominance of tests with $\alpha \approx \beta$ reflects the fact that departures from normality occur simultaneously in both tails. In such cases, balanced tail-weighting strategies are more effective, which explains the strong performance of such tests as $AD_{1,1}$, $AD_{0.9,0.9}$, and the classical CVM test.

7. Real data examples

In this section, we present the application of the analysed tests in real datasets to illustrate their potentiality. For the real data examples, the distribution of each test statistic under the null hypothesis of normality was obtained through a Monte Carlo simulation using the same critical values as those computed in the power study for the corresponding sample size. The details related to examples I–VII are presented in Table 14.

Table 14. Real data examples with sources, sample size, skewness and excess kurtosis values

Ex	Description	Source	n	γ_1	$\bar{\gamma}_2$
I	Socio-economic data (percentage of males involved in agriculture as occupation) for 47 French-speaking provinces of Switzerland.	R package swiss[2]	47	-0.331	-0.793
II	Measurements of the diameter of timber in 31 felled black cherry trees. The diameter is measured at 4 ft 6 in above the ground.	trees[1]	31	0.526	-0.556
III	The effect of vitamin C on tooth length in guinea pigs. Each animal received one of the three different doses of vitamin C (0.5, 1, and 2 mg/day) by one of the two alternative delivery methods, i.e. orange juice or ascorbic acid.	ToothGrowth[1]	60	-0.146	-0.976
IV	Data extracted from the Motor Trend US magazine showing fuel consumption for 32 automobiles.	mtcars[1]	32	0.640	-0.200
V	Lawyers' ratings of state judges in the US Superior Court (sound written rulings).	USJudgeRatings[1]	43	-0.699	0.030
VI	Arrests per 100,000 residents for rape in each of the 50 US states in 1973.	USArrests[4]	50	0.777	0.202
VII	Average height for American women aged 30–39	women[1]	15	0	-1.211

Source: authors' work.

When fitting the normal distribution to the data, we calculate the p-values for the analysed GoFTs based on 10^5 statistic values (see Table 15). The lowest p-values for all the analysed tests are in bold. The lowest p-values for the MAD tests are underlined. Non-normality is the most pronounced by the $AD_{0,1}$ (examples I, III, V) and $AD_{1,0}$ tests (examples II, IV). The lowest p-values for the MAD tests are observed for the $AD_{1,0}$ and $AD_{0,0}$tests (examples VI, VII). The obtained results are consistent with the simulation results, according to which, if the real data are negatively skewed, the $AD_{0,1}$ test is powerful, and if the real data are positively skewed, the $AD_{1,0}$ is powerful, and if the real data are symmetric, the $AD_{0,0}$ test is powerful.

Table 15. The p-values for the GoFTs related to examples I–VII

No.	GoFT	I	II	III	IV	V	VI	VII
1	LF	0.233	0.110	0.167	0.208	0.324	0.331	0.996
2	CVM	0.210	0.043	0.092	0.155	0.164	0.119	0.946
3	SW	0.190	0.089	0.107	0.123	0.096	**0.025**	0.727
4	$\lvert AD^L \rvert$	0.489	0.499	0.495	0.497	0.165	0.495	0.513
5	$\lvert AD^U \rvert$	0.490	0.240	0.489	0.279	0.490	0.122	**0.512**
6	$\lvert AD \rvert$	0.328	0.456	0.544	0.246	0.365	0.150	0.987
7	AD^L	0.424	0.279	0.372	0.228	0.181	0.102	0.863
8	AD^U	0.215	0.072	0.083	0.104	0.146	0.093	0.904
9	AD	0.196	0.046	0.087	0.123	0.122	0.074	0.925
10	$\overline{AD}$	0.196	0.046	0.087	0.123	0.122	0.074	0.925
11	$\overline{\overline{AD}}$	0.196	0.046	0.087	0.123	0.122	0.074	0.925
12	ADR^L	0.194	0.040	0.103	0.164	0.125	0.079	0.919
13	ADR^U	0.213	0.062	0.084	0.110	0.147	0.094	0.918
14	ADL^L	0.437	0.208	0.293	0.291	0.111	0.150	0.781
15	ADL^U	0.302	0.271	0.337	0.138	0.229	0.062	0.779
16	ADL	0.424	0.279	0.372	0.228	0.181	0.102	0.863
17	ADM^L	0.268	0.100	0.183	0.225	**0.095**	0.086	0.851
18	ADM^U	0.259	0.152	0.179	0.103	0.171	0.045	0.848
19	$\boldsymbol{AD_{0,1}}$	**0.173**	0.058	**0.082**	0.160	**0.095**	0.101	0.920
20	$\boldsymbol{AD_{1,0}}$	0.225	**0.038**	0.093	**0.097**	0.161	0.055	0.920
21	$\boldsymbol{AD_{0,0}}$	0.192	0.045	0.084	0.123	0.123	0.074	0.916
22	$\boldsymbol{AD_{1,1}}$	0.201	0.048	0.090	0.123	0.122	0.073	0.933
23	$\boldsymbol{AD_{0.3,0.3}}$	0.195	0.046	0.086	0.123	0.122	0.074	0.922
24	$\boldsymbol{AD_{3/8,3/8}}$	0.195	0.046	0.086	0.123	0.122	0.074	0.923
25	$\boldsymbol{AD_{\frac{127}{400},\frac{127}{400}}}$	0.195	0.046	0.086	0.123	0.122	0.074	0.922
26	$\boldsymbol{AD_{0.1,0.1}}$	0.193	0.046	0.084	0.123	0.123	0.074	0.918
27	$\boldsymbol{AD_{0.9,0.9}}$	0.200	0.047	0.089	0.123	0.122	0.073	0.932
28	$\boldsymbol{AD_{0.9,0.1}}$	0.219	0.039	0.091	0.102	0.152	0.058	0.921
29	$\boldsymbol{AD_{0.1,0.9}}$	0.177	0.055	0.083	0.151	0.100	0.094	0.922

Source: authors' work.

8. Conclusions

The aims of the research presented in this article were fully achieved: the family of MAD tests was expanded, four new formulas for the EDF were proposed, a flexible family of alternatives was created, consisting of older and newer distributions, and the powers of 29 tests were compared paying attention to the similarity measure of the alternative to the normal distribution.

Similarly to the modified L tests (Sulewski, 2021, 2022a), the parametrisation of the EDF based on Bloom's formula also influenced the power of the MAD test.

Tests $|AD^L|$ (groups A, D, E, F, G), $|AD^U|$ (B, D, E, F, H), $|AD|$ (D, E, F, G, H), AD^L (D, E, F, H), ADL^L (A, D, E, F, G), ADL^U (B, D, E, F, H), ADM^L (A, D, E, G), ADM^U (D F, H) and ADM (D) do not meet the basic assumptions, i.e. the power increases as the sample size increases and decreases as the similarity measure (36) increases.

The power of the $\overline{AD}$ and $\overline{\overline{AD}}$ tests is the same for all the analysed cases.

The tests best detect samples from asymmetric distributions with positive excess kurtosis.

The power of the $AD_{\alpha,\beta}$ $(\alpha = \beta;\ \alpha, \beta \leq 1)$ tests is very similar for all groups of alternatives. It is noteworthy that the power of the $AD_{\alpha,\beta}$ $(\alpha \neq \beta)$ tests for the given parameter values dominate in all of the analysed alternatives and similarity measures. The $AD_{1,0}$ test is recommended for positively skewed alternatives and the $AD_{0,1}$ test for negatively skewed alternatives. The $AD_{1,1}$ and $AD_{0,0}$ tests are recommended for symmetric alternatives.

The $AD_{\alpha,\beta}$ test for the given alternative and similarity measure, achieves high power in 53% of the cases in group E, 41% of the cases in group G, 34% of the cases in group D, 31% of the cases in group F and in 28% of the cases in group A. A result of less than 25% applies to groups B, C and H.

The analysis of real datasets has led to the conclusion that the MAD test is indeed effective.

References

Afeez, B. M., Maxwell, O., Otekunrin, O. A., & Happiness, O.-I. (2018). Selection and Validation of Comparative Study of Normality Test. *American Journal of Mathematics and Statistics*, *8*(6), 190–201. https://doi.org/10.5923/j.ajms.20180806.05.

Ahmad, F., & Khan, R. A. (2015). Power Comparison of Various Normality Tests. *Pakistan Journal of Statistics and Operation Research*, *11*(3), 331–345. https://doi.org/10.18187/pjsor.v11i3.845.

Aliaga, A. M., Martínez-González, E., Cayón, L., Argüeso, F., Sanz, J. L., & Barreiro, R. B. (2003). Goodness-of-fit tests of Gaussianity: constraints on the cumulants of the MAXIMA data. *New Astronomy Reviews*, *47*(8–10), 821–826. https://doi.org/10.1016/j.newar.2003.07.010.

Anderson, T. W., & Darling, D. A. (1952). Asymptotic theory of certain ʻgoodness-of-fitʼ criteria based on stochastic processes. *The Annals of Mathematical Statistics*, *23*(2), 193–212. https://doi.org/10.1214/aoms/1177729437.

Arnastauskaitė, J., Ruzgas, T., & Bražėnas, M. (2021). An Exhaustive Power Comparison of Normality Tests. *Mathematics*, *9*(7), 1–20. https://doi.org/10.3390/math9070788.

Bayoud, H. A. (2021). Tests of normality: new test and comparative study. *Communications in Statistics – Simulation and Computation*, *50*(12), 4442–4463. https://doi.org/10.1080/03610918.2019.1643883.

Bloom, G. (1958). *Statistical Estimates and Transformed Beta-Variables*. John Wiley & Sons. https://gwern.net/doc/statistics/order/1958-blom-orderstatistics.pdf.

Bonett, D. G., & Seier, E. (2002). A test of normality with high uniform power. *Computational Statistics & Data Analysis*, *40*(3), 435–445. https://doi.org/10.1016/S0167-9473(02)00074-9.

Bontemps, C., & Meddahi, N. (2005). Testing normality: a GMM approach. *Journal of Econometrics*, *124*(1), 149–186. https://doi.org/10.1016/j.jeconom.2004.02.014.

Borrajo, M. I., González-Manteiga, W., & Martínez-Miranda, M. D. (2024). Goodness-of-fit test for point processes first-order intensity. *Computational Statistics & Data Analysis*, *194*, 1–16. https://doi.org/10.1016/j.csda.2024.107929.

Brys, G., Hubert, M., & Struyf, A. (2008). Goodness-of-fit tests based on a robust measure of skewness. *Computational Statistics*, *23*(3), 429–442. https://doi.org/10.1007/s00180-007-0083-7.

Chernobai, A., Rachev, S. T., & Fabozzi, F. J. (2005). Composite goodness-of-fit tests for left-truncated loss samples. In C. F. Lee & J. C. Lee (Eds.), *Handbook of Financial Econometrics and Statistics* (pp. 575–596). Springer. https://doi.org/10.1007/978-1-4614-7750-1_20.

Coin, D. (2008). A goodness-of-fit test for normality based on polynomial regression. *Computational Statistics and Data Analysis*, *52*(4), 2185–2198. https://doi.org/10.1016/j.csda.2007.07.012.

Cramér, H. (1928). On the Composition of Elementary Errors. *Scandinavian Actuarial Journal*, (1), 13–74. https://doi.org/10.1080/03461238.1928.10416862.

Desgagné, A., & Lafaye de Micheaux, P. (2018). A powerful and interpretable alternative to the Jarque-Bera test of normality based on 2nd-power skewness and kurtosis, using the Rao's Score Test on the APD family. *Journal of Applied Statistics*, *45*(13), 2307–2327. https://doi.org/10.1080/02664763.2017.1415311.

Desgagné, A., Lafaye de Micheaux, P., & Ouimet, F. (2023). Goodness-of-fit tests for Laplace, Gaussian and exponential power distributions based on λ-th power skewness and kurtosis. *Statistics. A Journal of Theoretical and Applied Statistics*, *57*(1), 94–122. https://doi.org/10.1080/02331888.2022.2144859.

Esteban, M. D., Castellanos, M. E., Morales, D., & Vajda, I. (2001). Monte Carlo comparison of four normality tests using different entropy estimates. *Communications in Statistics – Simulation and Computation*, *30*(4), 761–785. https://doi.org/10.1081/SAC-100107780.

Gan, F. F., & Koehler, K. J. (1990). Goodness-of-fit tests based on P-P probability plots. *Technometrics*, *32*(3), 289–303. https://doi.org/10.2307/1269106.

Gel, Y. R., & Gastwirth, J. L. (2008). A Robust Modification of the Jarque-Bera Test of Normality. *Economics Letters*, *99*(1), 30–32. https://doi.org/10.1016/j.econlet.2007.05.022.

Gel, Y. R., Miao, W., & Gastwirth, J. L. (2007). Robust Directed Tests of Normality against Heavy-tailed Alternatives. *Computational Statistics & Data Analysis*, *51*(5), 2734–2746. https://doi.org/10.1016/j.csda.2006.08.022.

Giles, D. E. (2024). New Goodness-of-Fit Tests for the Kumaraswamy Distribution. *Stats*, *7*(2), 373–388. https://doi.org/10.3390/stats7020023.

Feuerverger, A. (2016). On Goodness of Fit for Operational Risk. *International Statistical Review*, *84*(3), 434–455. https://doi.org/10.1111/insr.12112.

Filliben, J. J. (1975). The probability plot correlation coefficient test for normality. *Technometrics*, *17*(1), 111–117. https://doi.org/10.1080/00401706.1975.10489279.

Harter, H. L., Khamis, H. J., & Lamb, R. E. (1984). Modified Kolmogorov-Smirnov tests of goodness of fit. *Communications in Statistics – Simulation and Computation*, *13*(3), 293–323. https://doi.org/10.1080/03610918408812378.

Hernandez, H. (2021). Testing for Normality: What is the Best Method?. *ForsChem Research Reports*, *6*, 101–138. https://doi.org/10.13140/RG.2.2.13926.14406.

Kellner, J., & Celisse, A. (2019). A one-sample test for normality with kernel methods. *Bernoulli*, *25*(3), 1816–1837. https://doi.org/10.3150/18-BEJ1037.

Kendall, M. G., & Stuart, A. (1968). *The advanced theory of statistics: Vol. 1. Distribution theory*. Hafner Publishing Company.

Khamis, H. J. (1990). The δ-corrected Kolmogorov-Smirnov test for goodness-of-fit. *Journal of Statistical Planning and Inference*, *24*(3), 317–335. https://doi.org/10.1016/0378-3758(90)90051-U.

Khamis, H. J. (1992). The δ-corrected Kolmogorov-Smirnov test with estimated parameters. *Journal of Nonparametric Statistics*, *2*(1), 17–27. https://doi.org/10.1080/10485259208832539.

Khamis, H. J. (1993). A comparative study of the d-corrected Kolmogorov-Smirnov test. *Journal of Applied Statistics*, *20*(3), 401–421. https://doi.org/10.1080/02664769300000040.

Khatun, N. (2021). Applications of normality test in statistical analysis. *Open Journal of Statistics*, *11*(1), 113–122.

Kolmogoroff, A. (1933). Sulla determinazione empirica di una legge di distribuzione. *Giornale dell'Istituto Italiano degli Attuari*, (4), 83–91.

Krauczi, E. (2009). A study of the quantile correlation test of normality. *TEST*, *18*(1), 156–165. https://doi.org/10.1007/s11749-007-0074-6.

Kuiper, N. H. (1960). Tests concerning random points on a circle. *Proceedings Series A. Indagationes Mathematicae*, *63*, 38–47. https://doi.org/10.1016/S1385-7258(60)50006-0.

Lilliefors, H. W. (1967). On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown. *Journal of the American Statistical Association, 62*(318), 399–402. https://doi.org/10.1080/01621459.1967.10482916.

Luceño, A. (2006). Fitting the generalized Pareto distribution to data using maximum goodness-of-fit estimators. *Computational Statistics & Data Analysis*, *51*(2), 904–917. https://doi.org/10.1016/j.csda.2005.09.011.

Ma, Y., Kitani, M., & Murakami, H. (2024). On modified Anderson-Darling test statistics with asymptotic properties. *Communications in Statistics – Theory and Methods*, *53*(4), 1420–1439. https://doi.org/10.1080/03610926.2022.2101121.

Malachov, A. N. (1978). Kumulantnyj analiz słuczajnych niegaussowych processow i ich prieobrazowanij. *Soviet Radio*.

Marange, C. S., & Qin, Y. (2019). An Empirical Likelihood Ratio Based Comparative Study on Tests for Normality of Residuals in Linear Models. *Metodološki Zvezki*, *16*(1), 1–16. https://doi.org/10.51936/ramh7128.

Mbah, A. K., & Paothong, A. (2015). Shapiro-Francia test compared to other normality test using expected *p*-value. *Journal of Statistical Computation and Simulation*, *85*(15), 3002–3016. https://doi.org/10.1080/00949655.2014.947986.

von Mises, R. (1931). Über einige Abschätzungen von Erwartungswerten. *Journal für die reine und angewandte Mathematik*, 1931(165), 184–193. https://doi.org/10.1515/crll.1931.165.184.Mishra, P., Pandey, C. M., Singh, U., Gupta, A., Sahu, C., & Keshri, A. (2019). Descriptive Statistics and Normality Tests for Statistical data. *Annals of Cardiac Anaesthesia*, *22*(1), 67–72. https://doi.org/10.4103/aca.ACA_157_18.

Nosakhare, U. H., & Bright, A. F. (2017). Evaluation of Techniques for Univariate Normality Test Using Monte Carlo Simulation. *American Journal of Theoretical and Applied Statistics*, *6*(5–1), 51–61. https://doi.org/10.11648/j.ajtas.s.2017060501.18.

Noughabi, H. A., & Arghami, N. R. (2011). Monte Carlo comparison of seven normality tests. *Journal of Statistical Computation and Simulation*, *81*(8), 965–972. https://doi.org/10.1080/00949650903580047.

Pearson, K. (1895). Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 186, 343–414.

Razali, N. M., & Wah, Y. B. (2011). Power comparisons of Shapiro-Wilk, Kolmogorov-Smirnov, Lilliefors and Anderson-Darling tests. *Journal of Statistical Modeling and Analytics*, *2*(1), 21–33.

Rodriguez, J. C., Viollaz, A. J. (1995). A Cramér-von Mises type goodness of fit test with asymmetric weight function. *Communications in Statistics – Theory and Methods*, *24*(4), 1095–1120. https://doi.org/10.1080/03610929508831542.

Romão, X., Delgado, R., & Costa, A. (2010). An empirical power comparison of univariate goodness-of-fit tests of normality. *Journal of Statistical Computation and Simulation*, *80*(5), 545–591. https://doi.org/10.1080/00949650902740824.

Shapiro, S. S., & Wilk, M. B. (1965). An analysis of variance test for normality. *Biometrika*, *52*(3/4), 591–611. https://doi.org/10.1093/biomet/52.3-4.591.

Sinclair, C. D., Spurr, B. D., & Ahmad, M. I. (1990). Modified Anderson darling test. *Communications in Statistics – Theory and Methods*, *19*(10), 3677–3686. https://doi.org/10.1080/03610929008830405.

Smirnov, N. (1948). Table for estimating the goodness of fit of empirical distributions. *The Annals of Mathematical Statistics*, *19*(2), 279–281. https://doi.org/10.1214/aoms/1177730256.

Stephens, M. A. (1974). EDF Statistics for Goodness of Fit and Some Comparisons. *Journal of the American Statistical Association*, *69*(347), 730–737. https://doi.org/10.1080/01621459.1974.10480196.

Stephens, M. A. (1979). *The Anderson-Darling statistic*. https://apps.dtic.mil/sti/tr/pdf/ADA079807.pdf.

Sulewski, P. (2019). Modification of Anderson-Darling goodness-of-fit test for normality. *Afinidad*, *76*(588), 270–277. https://www.raco.cat/index.php/afinidad/article/view/361876.

Sulewski, P. (2021). Two component modified Lilliefors test for normality. Equilibrium. *Equilibrium. Quarterly Journal of Economics and Economic Policy*, *16*(2), 429–455. https://doi.org/10.24136/eq.2021.016.

Sulewski, P. (2022a). Modified Lilliefors goodness-of-fit test for normality. *Communications in Statistics – Simulation and Computation*, *51*(3), 1199–1219. https://doi.org/10.1080/03610918.2019.1664580.

Sulewski, P. (2022b). Normal distribution with plasticizing component. *Communications in Statistics – Theory and Methods*, *51*(11), 3806–3835. https://doi.org/10.1080/03610926.2020.1837881.

Sulewski, P., & Stoltmann, D. (2026). Parameterized Kolmogorov–Smirnov Test for Normality. *Applied Sciences*, *16*(1), 1–44. https://doi.org/10.3390/app16010366.

Tavakoli, M., Arghami, N., & Abbasnejad, M. (2019). A goodness of Fit Test For Normality Based on Balakrishnan-Sanghvi Information. *Journal of The Iranian Statistical Society*, *18*(1), 177–190. https://doi.org/10.29252/jirss.18.1.177.

Terán-García, E., & Pérez-Fernández, R. (2024). A robust alternative to the Lilliefors test of normality. *Journal of Statistical Computation and Simulation*, *94*(7), 1–19. https://doi.org/10.1080/00949655.2023.2291138.

Torabi, H., Montazeri, N. H., & Grané, A. (2016). A test of normality based on the empirical distribution function. *SORT. Statistics and Operations Research Transactions*, *40*(1), 55–88. https://doi.org/10.2436/20.8080.02.35.

Uhm, T., & Yi, S. (2023). A comparison of normality testing methods by empirical power and distribution of *P*-values. *Communications in Statistics – Simulation and Computation*, *52*(9), 4445–4458. https://doi.org/10.1080/03610918.2021.1963450.

Uyanto, S. S. (2022). An Extensive Comparisons of 50 Univariate Goodness-of-fit Tests for Normality. *Austrian Journal of Statistics*, *51*(3), 45–97. https://doi.org/10.17713/ajs.v51i3.1279.

Watson, G. S. (1962). Goodness-of-fit tests on a circle. II. *Biometrika*, *49*(1–2), 57–63. https://doi.org/10.1093/biomet/49.1-2.57.

Wijekularathna, D. K., Manage, A. B., & Scariano, S. M. (2020). Power analysis of several normality tests: A Monte Carlo simulation study. *Communications in Statistics – Simulation and Computation*, *51*(3), 757–773. https://doi.org/10.1080/03610918.2019.1658780.

Yap, B. W., & Sim, C. H. (2011). Comparisons of various types of normality tests. *Journal of Statistical Computation and Simulation*, *81*(12), 2141–2155. https://doi.org/10.1080/00949655.2010.520163.

Yazici, B., & Yolacan, S. (2007). Comparison of various tests of normality. *Journal of Statistical Computation and Simulation*, *77*(2), 175–183. https://doi.org/10.1080/10629360600678310.

Appendix

Edgeworth series distribution

The PDF of the Edgeworth series (ES) with parameters γ_1 and $\bar{\gamma}_2$ is given by

$$f_{ES}(x;\gamma_{1,},\bar{\gamma}_2) = \phi(x;0,1)\left(1+\frac{1}{3!}\gamma_{1,}(x^3-3x)+\frac{1}{4!}\bar{\gamma}_2(x^4-6x^2+3)\right)\ (x \in R),$$

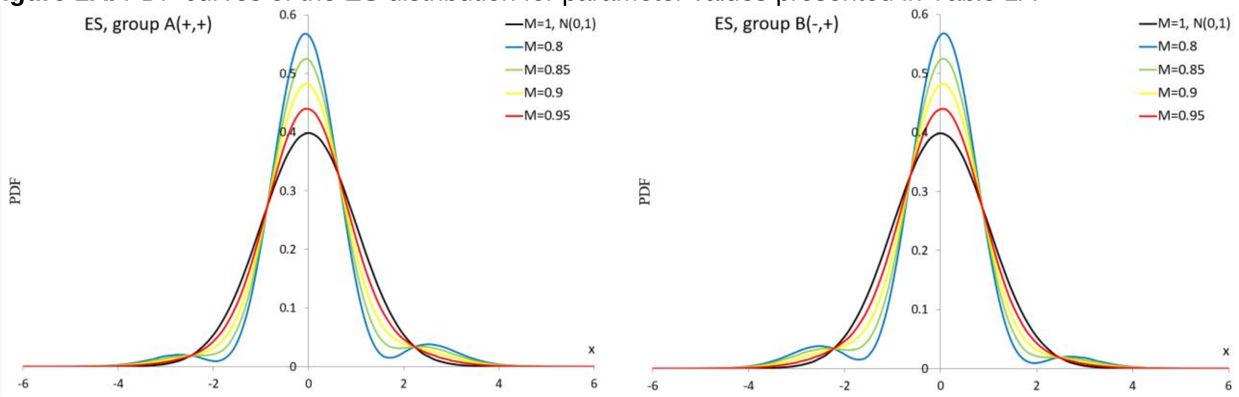
where $\gamma_{1,} \in R, \bar{\gamma}_2 \geq -2$.

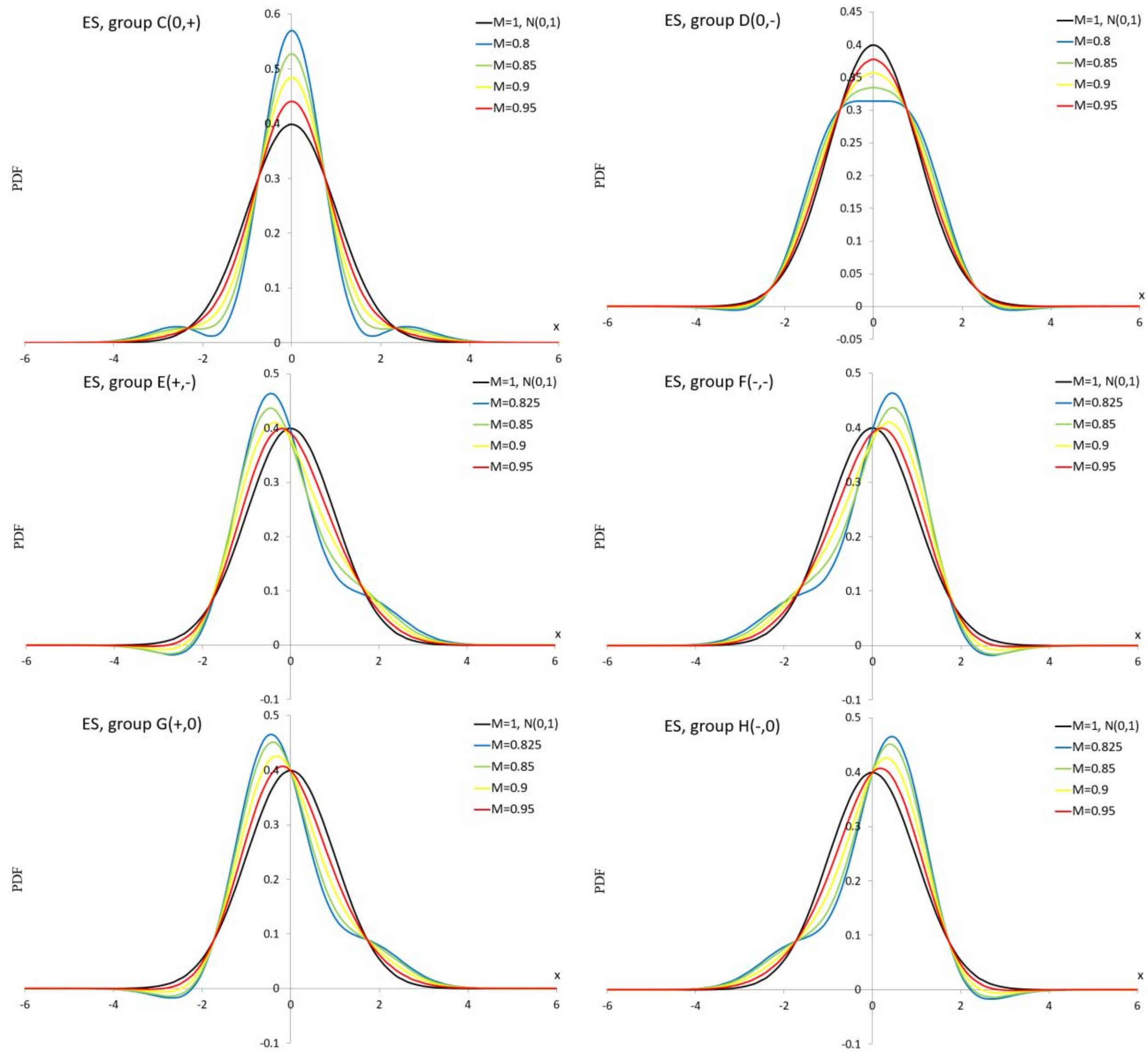
Table 1A. Vectors of the ES parameter **θ**, mean μ_a, standard deviation σ_a, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M. Groups O, A–H

Group	$\boldsymbol{\theta} = (\gamma_1, \bar{\gamma}_2)$	μ_a	σ_a	$\gamma_{1,}$	$\bar{\gamma}_2$	$M(\boldsymbol{\theta}; \mu, \sigma)$
O	(0,0)	0	1	0	0	$M(\boldsymbol{\theta}; 0,1) = 1$
A	0.4,3.33	0	1	0.4	3.33	$M(\boldsymbol{\theta}; 0,1) = 0.8$
	0.3,2.499	0	1	0.3	2.499	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0.2,1.666	0	1	0.2	1.666	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.1,0.833	0	1	0.1	0.833	$M(\boldsymbol{\theta}; 0,1) = 0.95$
B	-0.4,3.33	0	1	-0.4	3.33	$M(\boldsymbol{\theta}; 0,1) = 0.8$
	-0.3,2.499	0	1	-0.3	2.499	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	-0.2,1.666	0	1	-0.2	1.666	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-0.1,0.833	0	1	-0.1	0.833	$M(\boldsymbol{\theta}; 0,1) = 0.95$
C	0,3.428	0	1	0	3.428	$M(\boldsymbol{\theta}; 0,1) = 0.8$
	0,2.571	0	1	0	2.571	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,1.71	0	1	0	1.71	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0,0.85	0	1	0	0.85	$M(\boldsymbol{\theta}; 0,1) = 0.95$
D	0,-3.428	0	1	0	-3.428	$M(\boldsymbol{\theta}; 0,1) = 0.8$
	0,-2.571	0	1	0	-2.571	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,-1.71	0	1	0	-1.71	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0,-0.85	0	1	0	-0.85	$M(\boldsymbol{\theta}; 0,1) = 0.95$
E	1.39,-0.067	0	1	1.39	-0.067	$M(\boldsymbol{\theta}; 0,1) = 0.825$
	1.175,-0.46	0	1	1.175	-0.46	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0.775,-0.408	0	1	0.775	-0.408	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.39,-0.15	0	1	0.39	-0.15	$M(\boldsymbol{\theta}; 0,1) = 0.95$
F	-1.39,-0.067	0	1	-1.39	-0.067	$M(\boldsymbol{\theta}; 0,1) = 0.825$
	-1.175,-0.46	0	1	-1.175	-0.46	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	-0.775,- 0.408	0	1	-0.775	-0.408	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-0.39,-0.15	0	1	-0.39	-0.15	$M(\boldsymbol{\theta}; 0,1) = 0.95$
G	1.391,0	0	1	1.391	0	$M(\boldsymbol{\theta}; 0,1) = 0.825$
	1.19,0	0	1	1.19	0	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0.795,0	0	1	0.795	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.4,0	0	1	0.4	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$
H	-1.391,0	0	1	-1.391	0	$M(\boldsymbol{\theta}; 0,1) = 0.825$
	-1.19,0	0	1	-1.19	0	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	-0.795,0	0	1	-0.795	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-0.4,0	0	1	-0.4	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$

Source: authors' work.

Figure 1A. PDF curves of the ES distribution for parameter values presented in Table 1A





Source: authors' work.

Pearson distribution

Let $a = \frac{2\bar{\gamma}_2 - 3\gamma_1^2}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, b = \frac{|\gamma_1|(\bar{\gamma}_2 + 6)}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, c = \frac{4\bar{\gamma}_2 - 3\gamma_1^2 + 12}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, \Delta = b^2 - 4ac$, then the PDF of the Pearson (P) distribution is given by

$$f_P(x; \gamma_1, \bar{\gamma}_2) = \begin{cases} \dfrac{exp\left[\dfrac{2ab - b}{a(2ax + b)}\right]}{C_1(2ax + b)^{1/a}} & \Delta = 0 \\ \dfrac{exp\left[\dfrac{b - 2ab}{a\sqrt{4ac - b^2}} tan^{-1}\left(\dfrac{2ax + b}{\sqrt{4ac - b^2}}\right)\right]}{C_2(ax^2 + bx + c)^{1/(2a)}} & \Delta < 0 \\ \dfrac{\left(\dfrac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right)^{\frac{b - 2ab}{2a\sqrt{b^2 - 4ac}}}}{C_3(ax^2 + bx + c)^{1/(2a)}} & \Delta > 0 \end{cases}$$

where $x \in R, \gamma_1 \in R, \bar{\gamma}_2 \geq -2$ and C_1, C_2, C_3 are normalising constants defined as

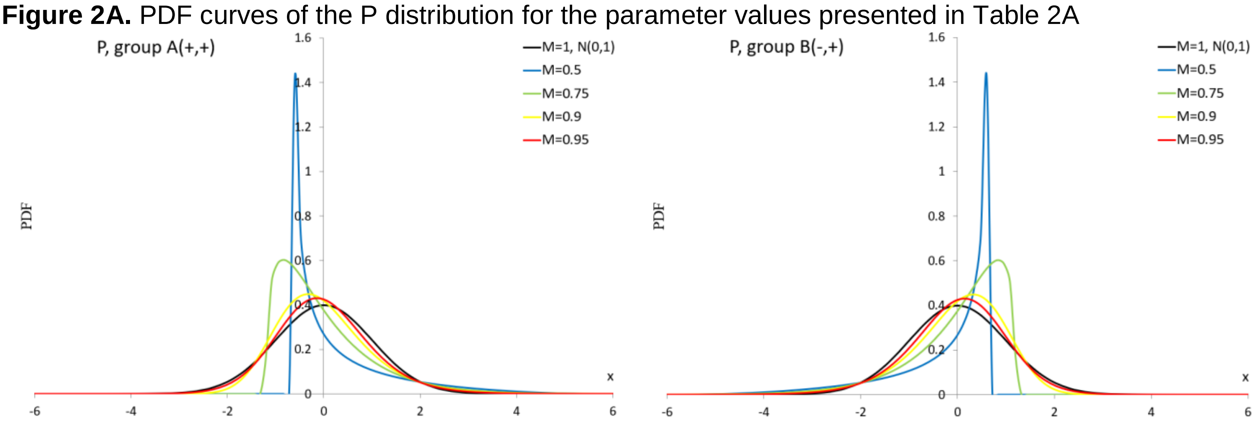
$$C_1 = \int_{-\infty}^{\infty} \frac{exp\left[\frac{2ab-b}{a(2ax+b)}\right]}{(2ax+b)^{1/a}} dx,\ C_2 = \int_{-\infty}^{\infty} \frac{exp\left[\frac{b-2ab}{a\sqrt{4ac-b^2}} tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{(ax^2+bx+c)^{1/(2a)}} dx,\ C_3 = \int_{-\infty}^{\infty} \frac{\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)^{\frac{b-2ab}{2a\sqrt{\Delta}}}}{C_8(ax^2+bx+c)^{1/(2a)}} dx.$$

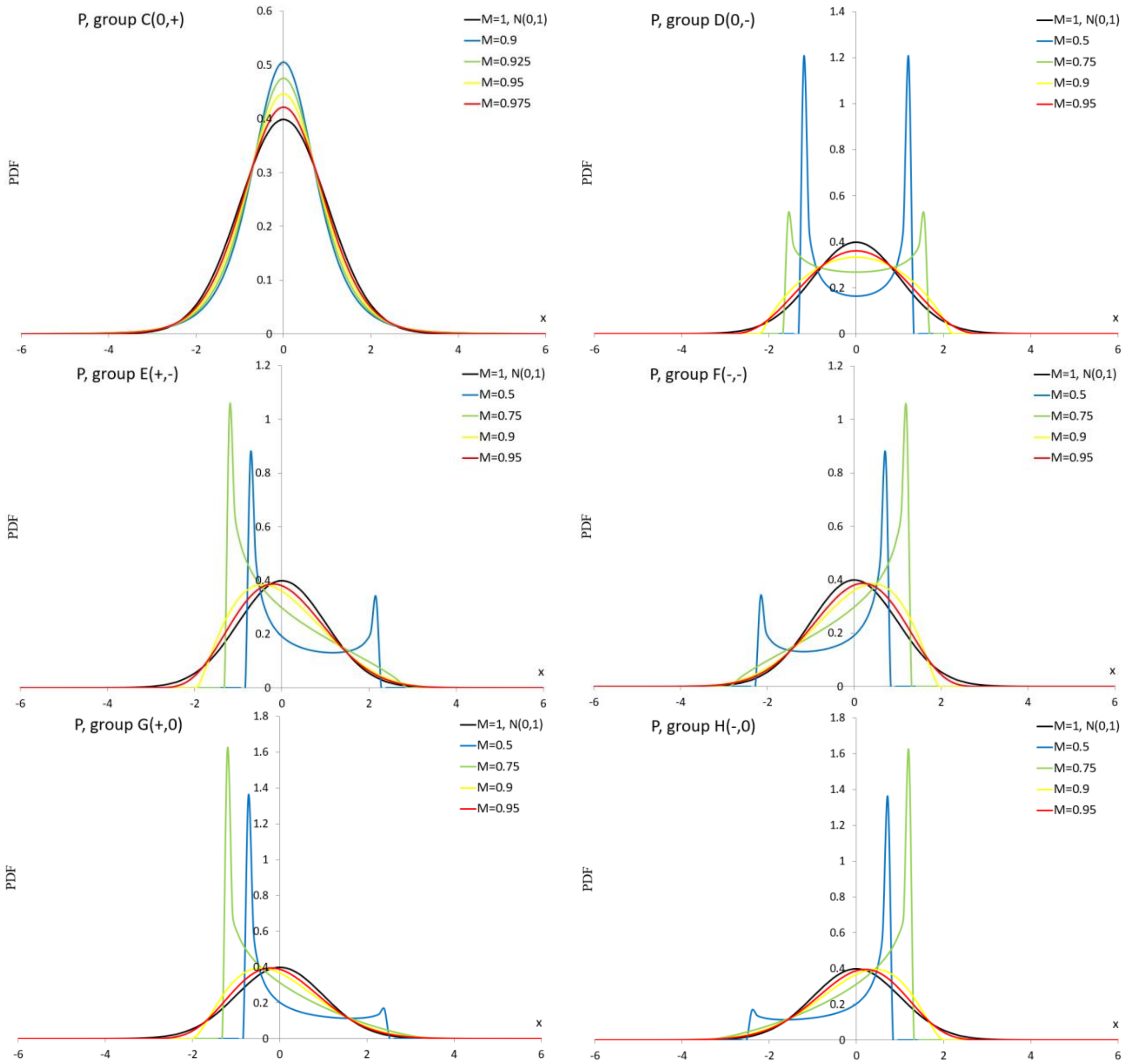
Table 2A. Vectors of the P parameter $\boldsymbol{\theta}$, mean μ_a, standard deviation σ_a, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M. Groups O, A-H

Group	$\boldsymbol{\theta} = (\gamma_1, \bar{\gamma}_2)$	μ_a	σ_a	γ_1	$\bar{\gamma}_2$	$M(\boldsymbol{\theta}; \mu, \sigma)$
O	(0,0)	0	1	0	0	$M(\boldsymbol{\theta}; 0,1) = 1$
A	(2.04,4.1)	0	1	2.04	4.1	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(1.62,3.845)	0	1	1.62	3.845	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(0.9,2)	0	1	0.9	2	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(0.4,0.94)	0	1	0.4	0.94	$M(\boldsymbol{\theta}; 0,1) = 0.95$
B	(-2.04,4.1)	0	1	-2.04	4.1	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(-1.62,3.845)	0	1	-1.62	3.845	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(-0.9,2)	0	1	-0.9	2	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(-0.4,0.94)	0	1	-0.4	0.94	$M(\boldsymbol{\theta}; 0,1) = 0.95$
C	(0,11.2)	0	1	0	11.2	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(0,3.65)	0	1	0	3.65	$M(\boldsymbol{\theta}; 0,1) = 0.925$
	(0,1.521)	0	1	0	1.521	$M(\boldsymbol{\theta}; 0,1) = 0.95$
	(0,0.55)	0	1	0	0.55	$M(\boldsymbol{\theta}; 0,1) = 0.975$
D	(0,-1.695)	0	1	0	-1.695	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(0,-1.315)	0	1	0	-1.315	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(0,-0.89)	0	1	0	-0.89	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(0,-0.588)	0	1	0	-0.588	$M(\boldsymbol{\theta}; 0,1) = 0.95$
E	(0.985,-0.5)	0	1	0.985	-0.5	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(0.715,-0.475)	0	1	0.715	-0.475	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(0.515,-0.2)	0	1	0.515	-0.2	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(0.315,-0.16)	0	1	0.315	-0.16	$M(\boldsymbol{\theta}; 0,1) = 0.95$
F	(-0.985,-0.5)	0	1	-0.985	-0.5	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(-0.715,-0.475)	0	1	-0.715	-0.475	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(-0.515,-0.2)	0	1	-0.515	-0.2	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(-0.315,-0.16)	0	1	-0.315	-0.16	$M(\boldsymbol{\theta}; 0,1) = 0.95$
G	(1.164,0)	0	1	1.164	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(0.879,0)	0	1	0.879	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(0.578,0)	0	1	0.578	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(0.354,0)	0	1	0.354	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$
H	(-1.164,0)	0	1	-1.164	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	(-0.879,0)	0	1	-0.879	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	(-0.578,0)	0	1	-0.578	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	(-0.354,0)	0	1	-0.354	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$

Source: authors' work.

Figure 2A. PDF curves of the P distribution for the parameter values presented in Table 2A





Source: authors' work.

Normal mixture distribution

The PDF of the normal mixture (NM) distribution is given by

$$f_{NM}(x;\mathbf{\theta}) = \omega\phi(x;\mu_1,\sigma_1) + (1-\omega)\phi(x;\mu_2,\sigma_2)\ (x \in R),$$

where $\mathbf{\theta} = (\mu_1, \sigma_1, \mu_2, \sigma_2, \omega)$ and $\mu_1, \mu_2 \in R, \sigma_1, \sigma_2 > 0, \omega \in [0,1]$.

Special cases of the NM distribution are:

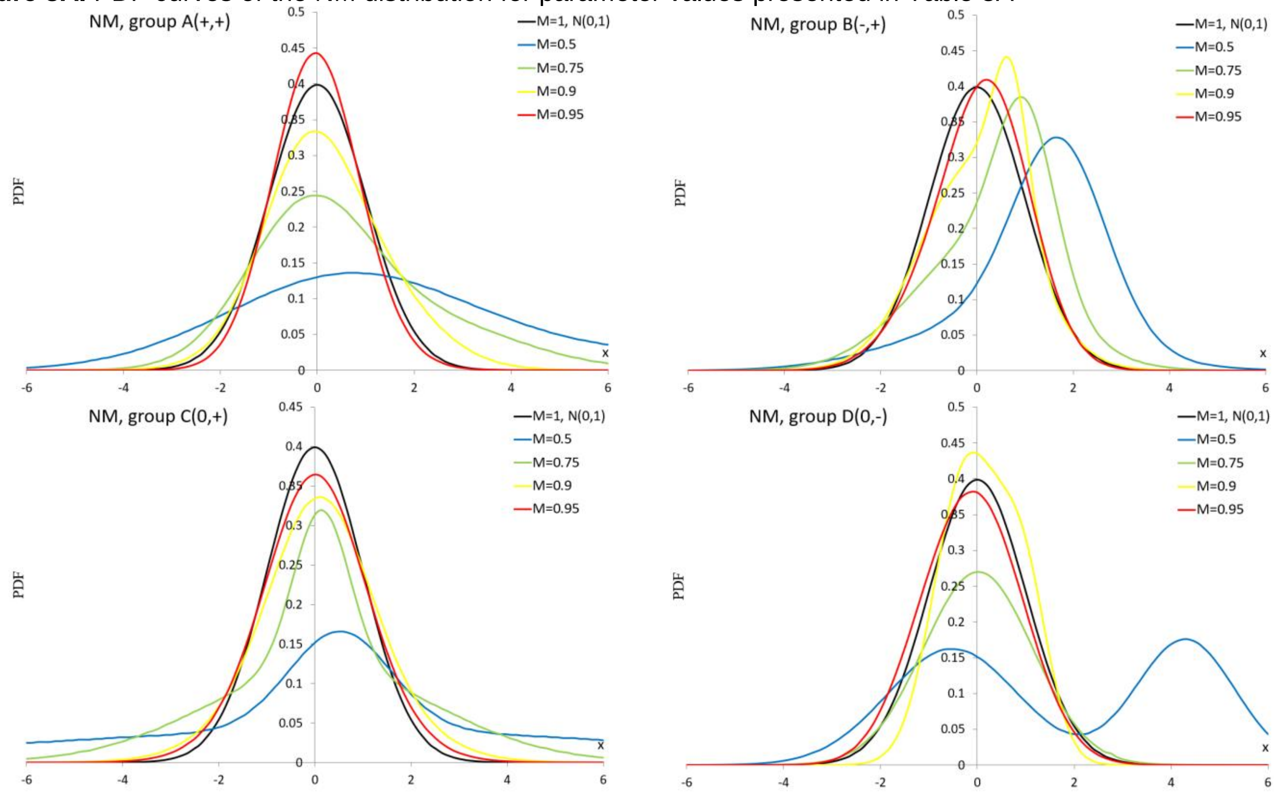
- normal $N(\mu_1, \sigma_1)$ for $\omega = 1$, $N(\mu_2, \sigma_2)$ for $\omega = 0$;
- location contaminated normal (LCN) $f_{LCM}(x;\mu_1,\omega) = f_{NM}(x;\mu_1,1,0,1,\omega)$;
- scale contaminated normal (SCN) $f_{SCN}(x;\ \sigma_1,\omega) = f_{NM}(x;0,\sigma_1,0,1,\omega)$.

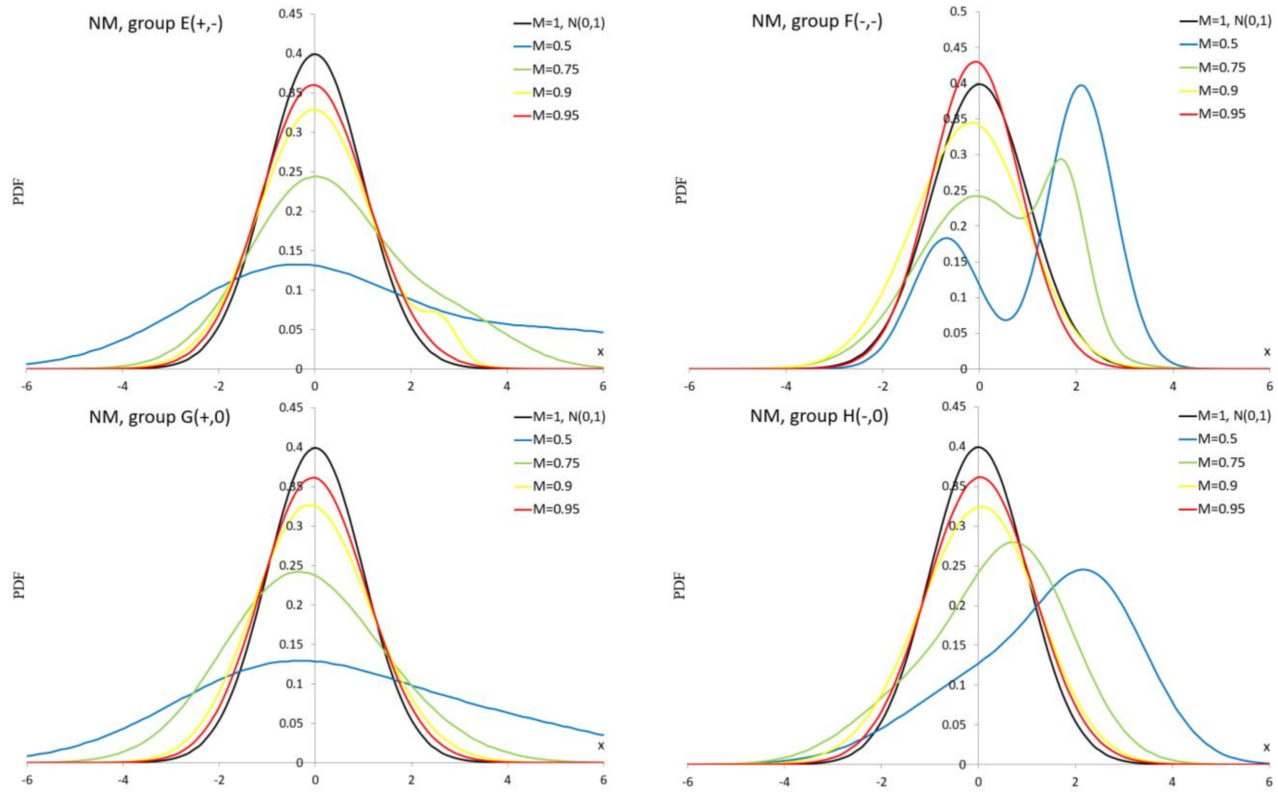
Table 3A. Vectors of the NM parameter $\mathbf{\theta}$, mean μ_a, standard deviation σ_a, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M. Groups O, A–H

Group	$\boldsymbol{\theta} = (\mu_1, \sigma_1, \mu_2, \sigma_2, \omega)$	μ_a	σ_a	$\gamma_{1,}$	$\bar{\gamma}_2$	$M(\boldsymbol{\theta}; \mu, \sigma)$
O	$(\mu_1, \sigma_1, \mu_2, \sigma_2, 1)$	0	1	0	0	$M(\boldsymbol{\theta}; \mu_1, \sigma_1) = 1$
	$(\mu_1, \sigma_1, \mu_2, \sigma_2, 0)$	0	1	0	0	$M(\boldsymbol{\theta}; \mu_2, \sigma_2) = 1$
A	0.572,2.472,5.614,3.454,0.787	1.646	3.408	0.685	0.755	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	-0.215,1.254,1.979,1.99,0.639	0.577	1.883	0.645	0.502	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.497,1.376,-0.268,0.884,0.612	0.2	1.265	0.287	0.249	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-0.098,0.857,0.31,1.007,0.767	-0.003	0.911	0.09	0.099	$M(\boldsymbol{\theta}; 0,1) = 0.95$
B	0.502,2.019,1.708,0.953,0.36	1.274	1.544	-0.748	1.502	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.06,1.437,1.004,0.609,0.634	0.406	1.285	-0.5	0.499	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.709,0.368,-0.072,1.115,0.193	0.079	1.06	-0.301	0.15	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.32,0.855,-0.873,0.923,0.782	0.06	1	-0.238	0.1	$M(\boldsymbol{\theta}; 0,1) = 0.95$
C	0.519,6.599,0.519,1.058,0.665	0.519	5.416	0	1.398	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.137,0.581,0.137,2.391,0.294	0.137	2.034	0	1.054	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.1,0.988,0.1,1.543,0.532	0.1	1.278	0	0.554	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.007,0.942,0.007,1.299,0.494	0.007	1.137	0	0.289	$M(\boldsymbol{\theta}; 0,1) = 0.95$
D	-0.511,1.353,4.293,1.021,0.551	1.645	2.681	0	-1.28	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	2.707,0.013,0.017,1.125,0.238	0.657	1.509	0	-1.001	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	1.243,0.621,-0.39,0.811,0.347	0.111	1.09	0	-0.63	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-1.112,0.794,0.023,0.974,0.13	0	0.897	0	-0.329	$M(\boldsymbol{\theta}; 0,1) = 0.95$
E	-0.475,2.22,5.318,2.427,0.721	1.141	3.457	0.5	-0.204	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	-0.019,1.369,2.979,1.15,0.829	0.494	1.748	0.339	-0.1	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	2.635,0.35,-0.015,1.166,0.038	0.086	1.253	0.137	-0.075	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	1.091,0.969,-0.111,1.056,0.1	0.009	1.108	0.05	-0.01	$M(\boldsymbol{\theta}; 0,1) = 0.95$
F	-0.692,0.705,2.1,0.679,0.324	1.195	1.476	-0.542	-0.852	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	-0.055,1.277,1.781,0.443,0.775	0.358	1.377	-0.3	-0.5	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	-0.09,1.08,-1.581,0.92,0.9	-0.239	1.155	-0.071	-0.042	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.386,0.845,-0.145,0.918,0.1	-0.092	0.925	-0.01	-0.011	$M(\boldsymbol{\theta}; 0,1) = 0.95$
G	2.686,3.099,-0.964,2.217,0.471	0.755	3.232	0.4	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	-0.56,1.465,1.411,1.45,0.8	-0.166	1.661	0.151	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	-0.286,1.114,0.984,1.105,0.801	-0.033	1.222	0.101	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-0.1,1.063,1.261,0.94,0.936	-0.013	1.106	0.053	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$
H	2.425,1.101,0.272,1.693,0.526	1.404	1.775	-0.499	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.864,1.125,-1.339,1.241,0.735	0.28	1.511	-0.386	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.429,1.078,-0.364,1.228,0.434	-0.02	1.23	-0.1	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.108,1.088,-0.524,1.073,0.879	0.032	1.106	-0.01	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$

Source: authors' work.

Figure 3A. PDF curves of the NM distribution for parameter values presented in Table 3A





Source: authors' work.

Normal distribution with plasticising component

The PDF of the normal distribution with plasticising component (NDPC) is given by

$$f_{NDPC}(x;\boldsymbol{\theta}) = \omega\phi(x;\mu_1,\sigma_1) + (1-\omega)\frac{c_2}{\sigma_2}\left|\frac{x-\mu_2}{\sigma_2}\right|^{c_2-1}\phi\left(\left|\frac{x-\mu_2}{\sigma_2}\right|^{c_2};0,1\right)\ (x \in R),$$

where $\boldsymbol{\theta} = (\mu_1,\sigma_1,\mu_2,\sigma_2,c_2,\omega)$ and $\mu_1,\mu_2 \in R, \sigma_1,\sigma_2 > 0, c_2 \geq 1, \omega \in [0,1]$.

Special cases of the NDPC distribution are:

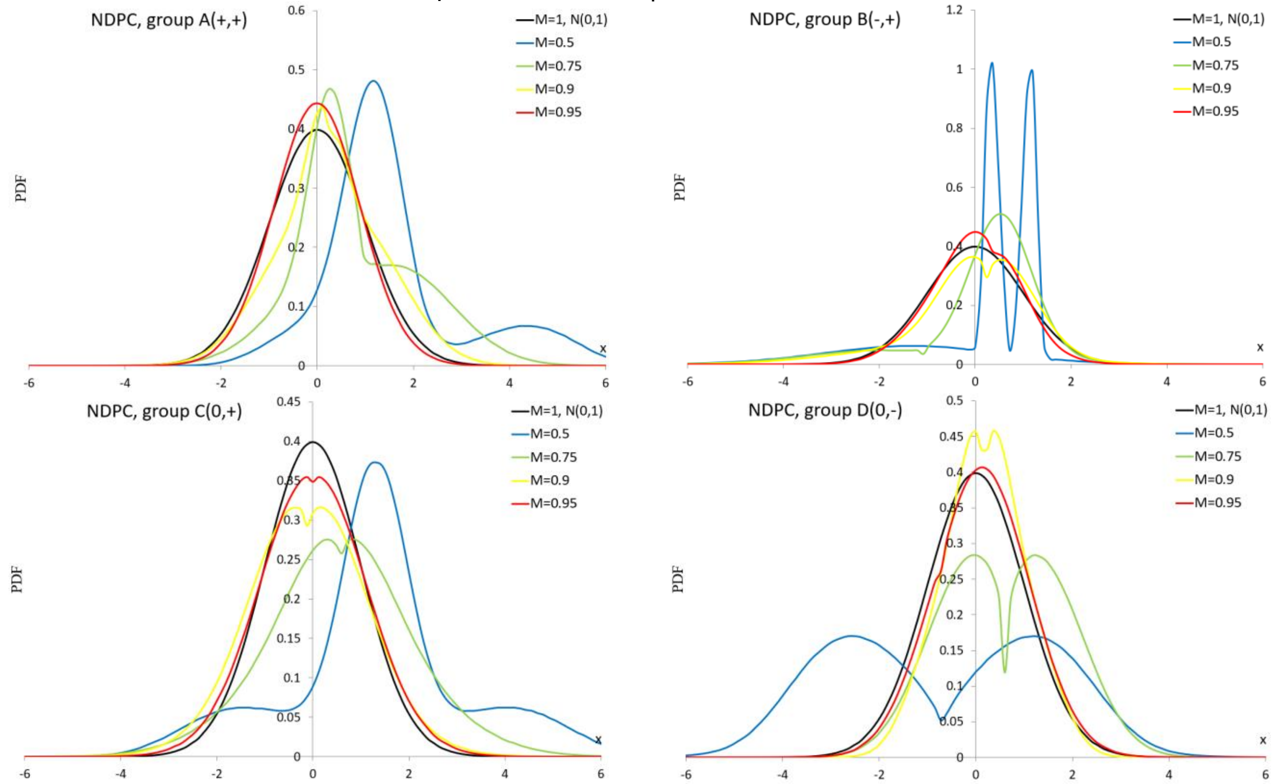
- $N(\mu_1,\sigma_1)$ for $\omega = 1$ and $N(\mu_2,\sigma_2)$ for $c_2 = 1, \omega = 0$;
- plasticising component (PC) $f_{PC}(x;\mu_2,\sigma_2,c_2)$ for $\omega = 0$.

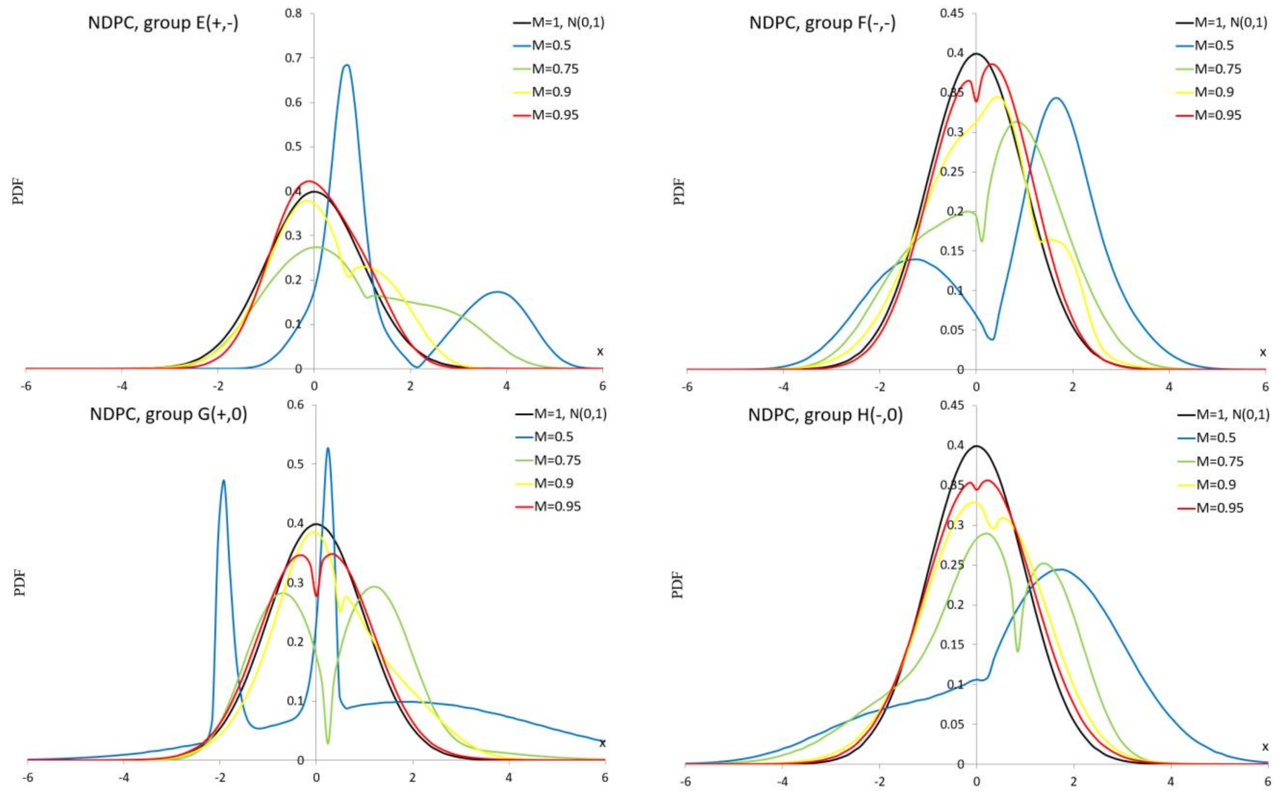
Table 4A. Vectors of the NDPC parameter $\boldsymbol{\theta}$, mean μ_a, standard deviation σ_a, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M. Groups O, A–H

Group	$\boldsymbol{\theta} = (\mu_1, \sigma_1, \mu_2, \sigma_2, c_2, \omega)$	μ_a	σ_a	$\gamma_{1,}$	$\bar{\gamma}_2$	$M(\boldsymbol{\theta}; \mu, \sigma)$
O	$\mu_1, \sigma_1, \mu_2, \sigma_2, c_2, 1$	0	1	0	0	$M(\boldsymbol{\theta}; \mu_1, \sigma_1) = 1$
	$\mu_1, \sigma_1, \mu_2, \sigma_2, 1{,}0$	0	1	0	0	$M(\boldsymbol{\theta}; \mu_2, \sigma_2) = 1$
A	1.194,0.601,2.186,2.592,2,0.666	1.526	1.5	1.002	1.001	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	0.265,0.415,0.996,1.541,1.16,0.313	0.767	1.288	0.426	0.152	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	0.173,0.358,0.289,1.268,1.132,0.198	0.266	1.104	0.056	0.071	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	0.047,1.02,-0.014,0.872,1,0.214	-0.001	0.906	0.012	0.06	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
B	-1.321,1.842,0.741,0.459,2.56,0.287	0.15	1.4	-1.764	3.3	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	0.539,0.632,-1.078,2.061,1.174,0.741	0.12	1.34	-1.499	2.986	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	-0.966,1.824,0.259,0.889,1.1,0.26	-0.059	1.305	-0.899	1.999	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	-0.099,0.938,0.399,0.646,1.204,0.831	-0.015	0.911	-0.125	0.036	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
C	1.308,0.656,1.308,3.261,2,0.613	1.308	1.884	0	0.504	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	0.571,1.023,0.571,1.962,1.15,0.505	0.571	1.508	0	0.325	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	-0.097,1.332,-0.097,1.058,1.1,0.614	-0.097	1.223	0	0.101	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	0.003,1.135,0.003,0.95,1.05,0.874	0.003	1.112	0	0.026	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
D	-0.692,2.203,-0.692,2.544,1.759,0.25	-0.692	2.265	0	-1	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	0.323,1.312,0.605,1.335,1.2,0.01	0.602	1.266	0	-0.587	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	0.179,0.494,0.179,1.163,1.426,0.443	0.179	0.862	0	-0.202	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	0.195,0.96,-0.719,0.858,1.109,0.918	0.12	0.983	0	-0.05	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
E	0.675,0.284,2.122,1.968,2.104,0.374	1.581	1.565	0.749	-0.849	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	0.423,1.032,1.058,2.077,1.815,0.494	0.744	1.544	0.311	-0.667	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	-0.134,0.993,0.671,1.211,1.479,0.583	0.202	1.115	0.115	-0.4	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	1.081,0.621,-0.216,0.755,1,0.24	0.095	0.912	0.1	-0.298	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
F	1.609,0.59,0.322,2.194,1.609,0.309	0.72	1.784	-0.491	-0.728	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	0.617,0.737,0.129,1.752,1.465,0.332	0.291	1.395	-0.239	-0.526	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	-0.046,1.156,1.261,0.799,1.87,0.876	0.116	1.191	-0.1	-0.2	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	0.155,0.882,0.019,1.184,1.175,0.581	0.098	0.995	-0.05	-0.188	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
G	1.88,2.736,-0.848,1.122,6.437,0.679	1.005	2.656	0.524	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	2.419,1.56,0.237,1.384,1.476,0.074	0.398	1.409	0.35	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	0.055,0.702,0.474,1.586,1.328,0.473	0.276	1.191	0.31	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	0.212,1.443,-0.012,1.057,1.088,0.1	0.01	1.079	0.05	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$
H	1.642,1.247,0.202,2.681,1.428,0.554	1	2.018	-0.594	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.5$
	-1.246,1.326,0.858,1.103,1.242,0.313	0.2	1.496	-0.5	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.75$
	-0.115,1.286,0.306,1.091,1.093,0.465	0.11	1.189	-0.1	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.9$
	0.084,0.949,0.03,1.214,1.047,0.423	0.053	1.098	-0.016	0	$M(\boldsymbol{\theta}; 0{,}1) = 0.95$

Source: authors' work.

Figure 4A. PDF curves of the NDPC for parameter values presented in Table 4A





Source: authors' work.

Plasticising component mixture distribution

The PDF of the plasticising component mixture distribution (PCM) is given by

$$f_{PCM}(x;\boldsymbol{\theta}) = \omega f_{PC}(x;\mu_1,\sigma_1,c_1) + (1-\omega) f_{PC}(x;\mu_2,\sigma_2,c_2)\ (x \in R),$$

where $f_{PC}(x;\mu,\sigma,c) = \frac{c}{\sigma}\left|\frac{x-\mu}{\sigma}\right|^{c-1} \phi\left(\left|\frac{x-\mu}{\sigma}\right|^{c};0,1\right)\ (x \in R)$ and $\boldsymbol{\theta} = (\mu_1,\sigma_1,c_1,\mu_2,\sigma_2,c_2,\omega)$, $\mu_1,\mu_2 \in R, \sigma_1,\sigma_2 > 0, c_1,c_2 \geq 1, \omega \in [0,1]$.

Special cases of the PCM distribution are:

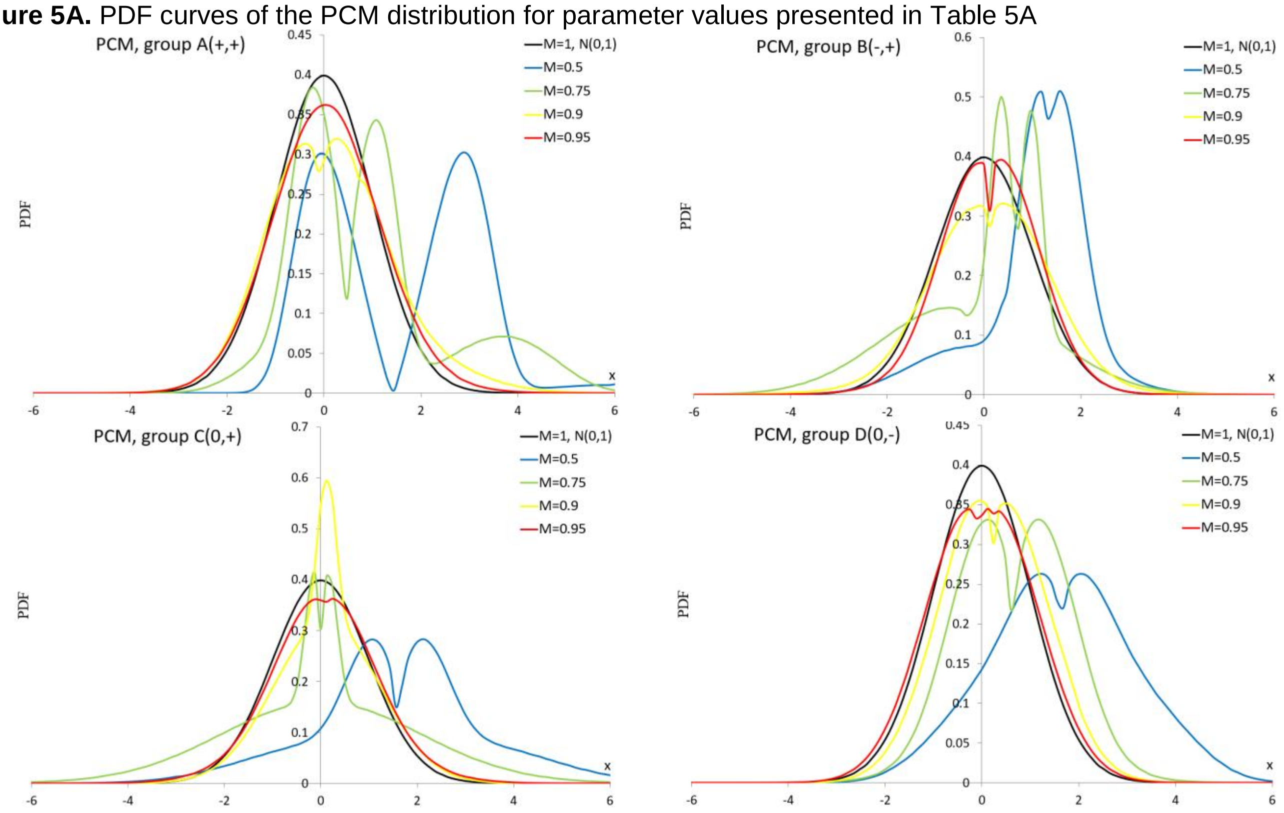
- $N(\mu_1,\sigma_1)$ for $c_1 = 1, \omega = 1$; $N(\mu_2,\sigma_2)$ for $c_2 = 1, \omega = 0$;
- plasticising component $PC(\mu_1,\sigma_1,c_1), PC(\mu_2,\sigma_2,c_2)$ for $\omega = 1, \omega = 0$, respectively.

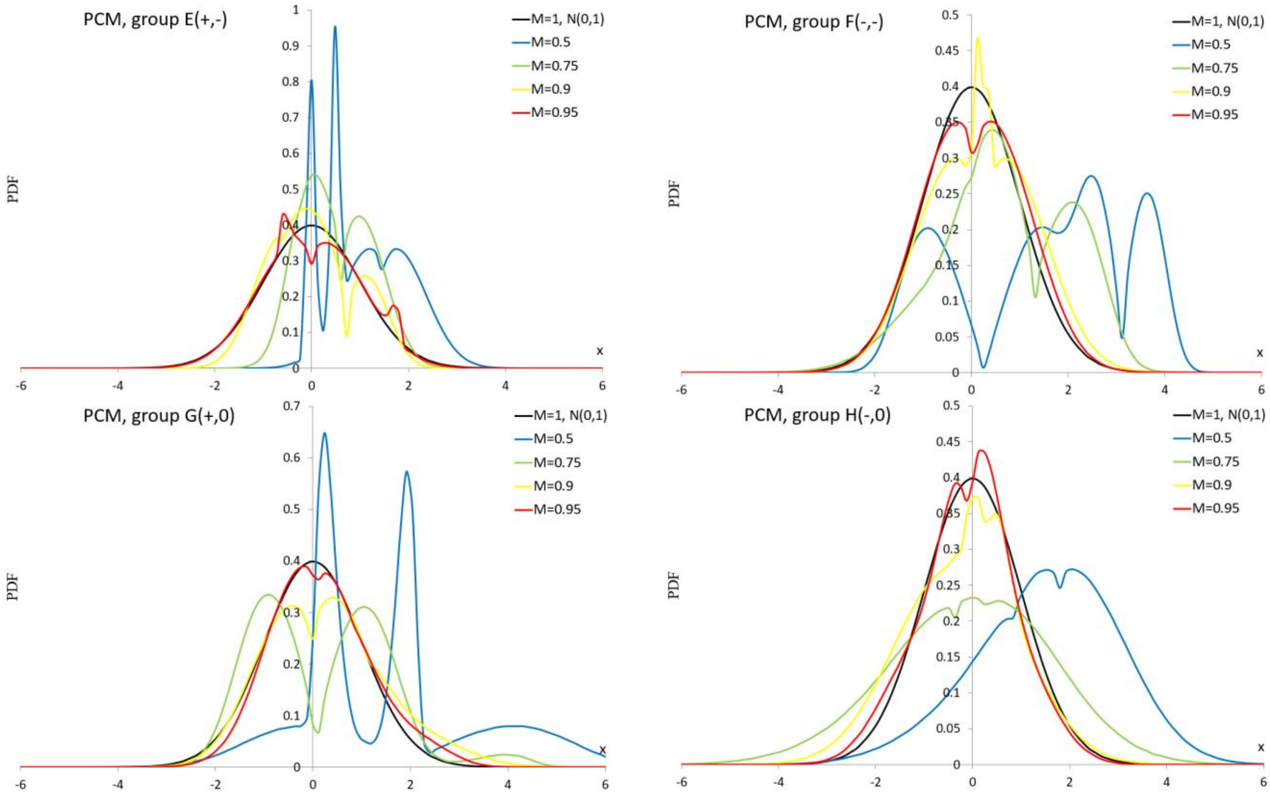
Table 5A. Vectors of the PCM parameter $\boldsymbol{\theta}$, mean μ_a, standard deviation σ_a, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M. Groups O, A–H

Group	$\boldsymbol{\theta} = (\mu_1, \sigma_1, c_1, \mu_2, \sigma_2, c_2, \omega)$	μ_a	σ_a	$\gamma_{1,}$	$\bar{\gamma}_2$	$M(\boldsymbol{\theta}; \mu, \sigma)$
O	$\mu_1, \sigma_1, 1, \mu_2, \sigma_2, c_2, 1$	0	1	0	0	$M(\boldsymbol{\theta}; \mu_1, \sigma_1) = 1$
	$\mu_1, \sigma_1, c_1, \mu_2, \sigma_2, 1, 0$	0	1	0	0	$M(\boldsymbol{\theta}; \mu_2, \sigma_2) = 1$
A	1.415,1.684,2.194,11.252,5.474,2.331,0.9	2.399	3.622	2.647	7.663	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.444,0.899,1.602,1.653,2.506,1.876,0.64	0.879	1.604	0.913	0.412	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	-0.076,1.056,1.1,0.701,1.646,1.095,0.71	0.149	1.268	0.374	0.374	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.026,1.078,1.001,0.701,1.646,1.174,0.95	0.06	1.117	0.099	0.148	$M(\boldsymbol{\theta}; 0,1) = 0.95$
B	1.366,0.572,1.11,0.502,1.669,1.253,0.658	1.071	1.099	-0.978	1.565	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.67,0.425,1.576,-0.323,1.696,1.05,0.349	0.024	1.444	-0.569	0.606	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	-0.204,2.209,1.205,0.133,1.139,1.05,0.076	0.107	1.224	-0.122	0.457	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.121,0.936,1.05,-0.17,1.917,1.411,0.95	0.106	0.982	-0.1	0.204	$M(\boldsymbol{\theta}; 0,1) = 0.95$
C	1.597,2.518,1.263,1.596,0.856,1.285,0.526	1.597	1.797	0	0.601	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.012,0.274,1.256,0.012,2.046,1.01,0.183	0.012	1.846	0	0.598	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.127,1.089,1.01,0.127,0.183,1.01,0.863	0.127	1.01	0	0.401	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.075,0.973,1.01,0.075,1.964,1.362,0.867	0.075	1.119	0	0.387	$M(\boldsymbol{\theta}; 0,1) = 0.95$
D	1.631,0.893,1.05,1.632,2.104,1.554,0.498	1.632	1.488	0	-0.268	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.639,1.576,1.167,0.64,1.085,1.199,0.163	0.64	1.12	0	-0.251	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.666,1.123,4.041,0.233,1.069,1.05,0.01	0.237	1.052	0	-0.198	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.225,1.087,1.05,-0.067,1.094,1.05,0.233	0.001	1.081	0	-0.18	$M(\boldsymbol{\theta}; 0,1) = 0.95$
E	1.472,0.782,1.11,0.236,0.291,3.203,0.692	1.091	0.861	0.38	-0.8	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	-0.196,0.341,1.064,0.613,0.758,1.204,0.153	0.489	0.734	0.201	-0.7	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.722,0.703,1.304,-0.57,0.598,1.05,0.455	0.018	0.893	0.179	-0.617	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.584,1.171,9.804,-0.016,1.024,1.076,0.05	0.014	1.013	0.028	-0.351	$M(\boldsymbol{\theta}; 0,1) = 0.95$
F	0.261,1.419,1.909,3.099,0.744,1.567,0.57	1.481	1.757	-0.3	-1.107	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0.037,1.295,1.076,1.316,1.171,1.654,0.485	0.696	1.326	-0.204	-0.4	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.201,0.121,1.573,0.184,1.177,1.161,0.066	0.185	1.087	-0.003	-0.331	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.049,1.063,1.088,1.392,0.511,1.05,0.99	0.062	1.038	-0.008	-0.328	$M(\boldsymbol{\theta}; 0,1) = 0.95$
G	1.088,0.894,3.782,1.969,2.71,1.792,0.55	1.484	1.793	0.6	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	1.515,2.553,3.55,0.07,1.328,1.619,0.07	0.171	1.359	0.501	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	-0.034,1.072,1.159,1.146,1.51,1.301,0.756	0.254	1.238	0.401	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	0.825,1.615,1.868,0.067,0.934,1.05,0.141	0.174	1.044	0.336	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$
H	0.816,1.867,1.24,1.787,1.272,1.05,0.278	1.517	1.475	-0.302	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	-0.364,1.889,1.057,0.29,1.413,1.05,0.527	-0.055	1.682	-0.154	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0.286,0.405,1.27,-0.263,1.261,1.05,0.112	-0.202	1.188	-0.128	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
	-0.153,1.344,1.349,-0.024,0.539,1.05,0.565	-0.097	1	-0.12	0	$M(\boldsymbol{\theta}; 0,1) = 0.95$

Source: authors' work.

Figure 5A. PDF curves of the PCM distribution for parameter values presented in Table 5A





Source: authors' work.